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On the open-closed B-model

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Abstract: We study the coupling of the closed string to the open string in the topological B-model. These couplings can be viewed as gauge invariant observables in the open string field theory, or as deformations of the differential graded algebra describing the OSFT. This is interpreted as an intertwining map from the closed string sector to the deformation (Hochschild) complex of the open string algebra. By an explicit calculation we show that this map induces an isomorphism of Gerstenhaber algebras on the level of cohomology. Reversely, this can be used to derive the closed string from the open string. We shortly comment on generalizations to other models, such as the A-model.

Keywords: Topological Field Theories, Chern-Simons Theories, D-branes, String Field Theory.

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1. Introduction

In this paper we will refine and work out in more detail the study of topological open and closed strings in [1], focusing on the topological B-model. The main questions we will study in this paper are the coupling of closed string to the open string sector, and how the closed string is encoded in the open string.

The question of where the closed string strings are in the open strings has been lingering in the context of string field theory [2]. Indeed poles corresponding to the closed string operators are present in the open string field theory [3, 4]. One way is to introduce them explicitly [5, 6], leading to open-closed string field theory. Another way is to find the closed string operators as observables in the open string [7, 8, 9]. These gauge invariant observables heuristically can be interpreted as integrals over a cycle of the open string field, $\Phi_C(A) = \int_C A$. Here A is the open string field, and C is induced by the closed string operator. In this paper we consider both approaches in a particularly simple topological string field theory toy model. The main idea for the latter approach is to view the open string field theory algebra as a noncommutative geometry [10], and interpret the closed string operators as the cycles C in this geometry.

In [1] we discussed mixed correlators for the open/closed topological string; see also [11, 12]. The open $(n+1)$-point functions $F_{a_0 a_1 \dots a_n}$, where the a_i label the open string operators, define structure constants for n-linear maps m_n. Together with the BRST operator, $Q = m_1$, they can be shown to form an A_∞ algebra [13, 1]. Especially, the 3-point functions F_{abc} are the structure constants of a product $m = m_2$.

The next step is to couple the boundary operators to operators in the bulk. This leads to mixed $(n+2)$-point functions which are defined through the correlators

$$\Phi_{I a_0 a_1 \dots a_n} = \pm \left\langle \hat{\phi}_I \hat{\alpha}_{a_0} \int \hat{\alpha}^{(1)}_{a_1} \cdots \int \hat{\alpha}^{(1)}_{a_n} \right\rangle , \tag{1.1}$$

involving a bulk operator $\hat{\phi}_I$ and boundary operators $\hat{\alpha}_a$. Introducing deformed structure constants $F_{abc}(t)$, where t^I are couplings for the closed string operators deforming the open string algebra, it follows from the Ward identities that

$$\partial_{t^I} F_{abc} = \Phi_{Iabc}\,, \tag{1.2}$$

expressing the fact that the mixed correlators are related to the deformation of the algebra of boundary operators. A similar relation is valid for the other mixed correlators and structure constants. Through the mixed correlators (1.1) we therefore have a natural map from the closed string operators to the deformations of the open string algebra.

It is a well known mathematical fact that the deformation of an A_∞ algebra is controlled by its Hochschild complex. This complex can be represented as the space of all multilinear maps on the algebra. In fact, we can interpret the mixed correlators (1.1) as multilinear maps on the open string algebra. This allows us to interpret the above mentioned map as a map from closed string operators to the Hochschild complex. Both the Hochschild complex and the closed string have a structure of L_∞ algebra. More generally, the corresponding cohomologies carry the structure of a Gerstenhaber algebra. In [1] we showed that the map from the closed string to the Hochschild complex intertwines these structures. Most importantly, the BRST operator in the closed string corresponds to the natural coboundary in the Hochschild complex. This implies that the map induces a natural map between the cohomologies. Moreover, this map intertwines the structure of a Gerstenhaber algebra. A natural question then is whether these Gerstenhaber algebras are actually the same.

An example of this is given by the problem of deformation quantization [14]–[18]. This studies the deformation of a commutative product by a Poisson Lie bracket to a full noncommutative star-product. In [19] it was shown that the solution to this problem given by Kontsevich [14, 15] can be interpreted in terms of similar correlation functions of a certain topological open string theory, as alluded to by Kontsevich [14]. The star product deformation leads to a noncommutative geometry [10]. In fact, this topological construction can be embedded in string theory, as was shown in [20, 21], first initialized in [22]. A purpose of [1] was to generalize the deformations of [19] to more general topological string theories.

In this paper we want to work out in detail the relation between the mixed correlators and the deformations of the A_∞ structure in more detail for the case of the B-model topological string [23, 24, 25].

The open string sector of the B-model is the holomorphic Chern-Simons theory first studied by Witten [26]. There it was also shown to be a string field theory. The critical points of this theory are holomorphic bundles on a Calabi-Yau manifold. The closed string B-model is the Kodaira-Spencer theory [24], the deformation theory of complex structures. Its extended space of observables is given by the Dolbeault cohomology, or more precisely $\bigoplus_{p,q} H^{-p,q}(M)$. We find this space precisely as the Hochschild cohomology of the open string algebra $H^*_{\bar{\partial}_A}(M, \mathrm{End}(E))$. In other words: we find the (on-shell) closed string algebra as the Hochschild cohomology of the open string algebra. We want to put this forward as a general conjecture, which becomes: The Hochschild cohomology of the open string field theory is isomorphic to the (on-shell) closed string algebra. As the Hochschild cohomology can be completely derived from the open string algebra, this implies a derivation of the closed string from the open string algebra. We want to see the considerations of the present paper as evidence for this conjecture.

The open and closed B-model has received some attention, mainly in the recent year [27]–[31]. This was mainly concerned with calculations of the mixed superpotential for non-compact Calabi-Yau manifolds. This leads in particular to interesting generalizations of mirror symmetry [32, 33] to the open string sector. A different type of relation between the open and closed string have been studied starting from [34].

The paper is organized as follows. In section 2 we give a short review of topological open-closed string theory, summarizing the results of [1], and adding some general remarks on gauge invariant observables. In section 3 we give a brief discussion of the B-model.

In section 4 we calculate the mixed correlation functions in the B-model with a single closed string insertion. In section 5 we interpret these correlators in terms of deformations of the open string algebra, and present the open-closed string field action (the superpotential) to first order in the closed string field.

In section 6 we calculate the Hochschild cohomology of the open string algebra. This calculation shows that there is an isomorphism between the two.

In section 7 the BV structure on the Hochschild cohomology is used to construct a BV sigma model. We show that this model gives an off-shell description of the (closed) B-model.

In section 8 we interpret our results in terms of cycles in a noncommutative geometry. We also comment on the more precise identification of the closed string with the Hochschild cohomology and the relation to cyclic cohomology.

In section 9 we shortly discuss the calculations of the Hochschild cohomology for other models, most prominently the A-model.

We end up with conclusion and some further discussions of our results in section 10.

1.1 Notation

In this paper, the lowercase Greek indices $\mu, \nu, \dots$ from the middle of the alphabet will denote holomorphic directions in the (complex) target space, while corresponding barred indices $\bar{\mu}, \bar{\nu}, \dots$ denote the anti-holomorphic directions. In the case of open strings, these will denote the direction along the brane; Latin indices $i, j, \dots$ and $\bar{\imath}, \bar{\jmath}, \dots$ will denote the (anti)holomorphic transverse directions. Latin indices $a, b, \dots$ from the beginning of the alphabet will number open string (boundary) operators, while the uppercase Latin indices $I, J, \dots$ from the middle of the alphabet number closed string operators. We will use hats to distinguish operators from the corresponding forms.

The holomorphic tangent bundle of a complex manifold M is denoted $\mathcal{T}_M$, while the anti-holomorphic tangent bundle is denoted $\overline{\mathcal{T}}_M$. Similarly, holomorphic and anti-holomorphic cotangent bundles are denoted $\mathcal{T}^*_M$ and $\overline{\mathcal{T}}^*_M$ respectively. Also $\mathcal{N}_C$ will denote the holomorphic normal bundle of any complex cycle C. The space of sections of the exterior algebra of (p, q)-forms $\bigwedge^p \mathcal{T}^*_M \otimes \bigwedge^q \overline{\mathcal{T}}^*_M$, is denoted $\Omega^{p,q}(M)$. The space of (p, q)-forms with values in some (holomorphic) bundle E is denoted $\Omega^{p,q}(M, E)$. We will call a

$(0, q)$-form with values in the pth exterior power of the holomorphic tangent space $\bigwedge^p \mathcal{T}_M$ a $(-p, q)$-form, and the space of such forms is denoted $\Omega^{-p,q}(M) \equiv \Omega^{0,q}(M, \bigwedge^p \mathcal{T}_M)$. Analogously, we use the notation $H^{-p,q}(M)$ for its $\bar{\partial}$-cohomology.

The space of multilinear maps on an algebra $\mathcal{A}$ (of order n) is denoted $C^n(\mathcal{A}, \mathcal{A}) = \mathrm{Hom}(\mathcal{A}^{\otimes n}, \mathcal{A})$. For a graded algebra $\mathcal{A}$ we denote by $\mathrm{Hom}(\mathcal{A}^{\otimes n}, \mathcal{A})^q$ the space of n-linear maps raising the total degree by q. This is also understood as a complex with the Hochschild differential δ_m. The cohomology of this complex is denoted $HH^*(\mathcal{A})$.[1] The Hochschild complex of a differential graded algebra $\mathcal{A}$ as a double complex is the same space with the two differentials δ_m and δ_Q, and is denoted $\mathrm{Hoch}(\mathcal{A})$. The total cohomology of this double complex is denoted $H^*(\mathrm{Hoch}(\mathcal{A}))$.

2. Deformations of topological open strings

In this section we repeat the basic results of [1], and relate the structure found there to gauge invariant observables from the point of view of string field theory.

2.1 Correlators and deformations

Central in topological field theories is the existence of a BRST operator Q such that the energy-momentum tensor $T_{\alpha\beta}$ is BRST exact. This implies that there must be a tensor current $b_{\alpha\beta}$ of ghost number -1 such that $\{Q, b_{\alpha\beta}\} = T_{\alpha\beta}$. The tensor $b_{\alpha\beta}$ is the current for a charge G, which is a 1-form of ghost number -1. It satisfies the anticommutation relation $\{Q, G\} = d$. The operator G is used to define the descendants of an operator $\hat{\alpha}$ recursively as $\hat{\alpha}^{(p+1)} = G\hat{\alpha}^{(p)}$. If $Q\hat{\alpha} = 0$, then the these are solutions to the descent equations $Q\hat{\alpha}^{(p+1)} = d\hat{\alpha}^{(p)}$.

Topological open strings can be characterized by the structure constants, which are defined by the correlation functions

$$F_{a_0 a_1 \ldots a_n} = (-1)^{\epsilon} \left\langle \hat{\alpha}_{a_0} \hat{\alpha}_{a_1} \hat{\alpha}_{a_2} \int \hat{\alpha}^{(1)}_{a_3} \cdots \int \hat{\alpha}^{(1)}_{a_n} \right\rangle . \tag{2.1}$$

where $\epsilon = n|\alpha_{a_1}| + \sum_{i \geq 2}(n-i)|\alpha_{a_i}|$. When we use the open string metric $g_{ab} = \langle \hat{\alpha}_a \hat{\alpha}_b \rangle$ to raise and lower indices, these can be interpreted as the structure constants of an A_∞ algebra formed by multi-linear operations m_n, as

$$m_n(\hat{\alpha}_{a_1}, \cdots, \hat{\alpha}_{a_n}) = F^{a_0}_{a_1 \ldots a_n} \hat{\alpha}_{a_0} . \tag{2.2}$$

For example, the 3-point functions F^a_{bc} are structure constants of the product $m = m_2$. Ward identities, similar to the WDVV equations [35, 36] for the closed string, assure that this product is associative on-shell. More generally, the multi-linear operations m_n satisfy higher associativity relations which are known as an A_∞ algebra [37, 38, 39, 13, 40, 41, 42]. In this paper we will assume for symplicity that $m_n = 0$ for $n \geq 3$, so that the undeformed open string theory is a genuine differential graded associative algebra. This is certainly correct for our main focus, the B-model.

[1] Usually, the Hochschild cohomology of multilinear maps with values in a bimodule $\mathcal{M}$ is denoted $HH^*(\mathcal{A}, \mathcal{M})$, but in this paper we only meet the case $\mathcal{M} = \mathcal{A}$, so we will not include this module in our notation.

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When we include (on-shell) bulk operators $\hat{\phi}_I$, the mixed correlators can be interpreted as deformations of the A_∞ algebra, by deforming the operations m_n. Introducing couplings t^I for the bulk operators, we write the deformed structure constants as $F_{a_0 a_1 \dots a_n}(t)$, and the deformed multilinear maps as $m_n(t)$. We find the following interpretation of the correlators with one bulk insertion

$$\Phi_I^{(n)}(\hat{\alpha}_{a_1}, \dots, \hat{\alpha}_{a_n}) = (-1)^\epsilon \left\langle \hat{\phi}_I \hat{\alpha}_b \int \hat{\alpha}_{a_1}^{(1)} \cdots \int \hat{\alpha}_{a_n}^{(1)} \right\rangle g^{bc} \hat{\alpha}_c \equiv \Phi^{(n)}_{I b a_1 \dots a_n} \hat{\alpha}^b = \partial_{t^I} F_{b a_1 \dots a_n}(0) \hat{\alpha}^b \,, \tag{2.3}$$

where $\epsilon = |\phi_i||\alpha_b| + \sum_{i \geq 1}(n-i)|\alpha_{a_i}|$. We interpret these correlators as the structure constants of a set of multilinear maps $\Phi_I^{(n)}$ defined by the second equality, which we collectively denote by Φ_I. So we can write $\Phi_I = \sum_n \Phi_I^{(n)} = \sum_n \partial_{t^I} m_n(0)$. The maps Φ_I can be viewed as elements of the Hochschild complex of multi-linear maps $C^*(\mathcal{A}, \mathcal{A}) = \bigoplus_n \mathrm{Hom}(\mathcal{A}^{\otimes n}, \mathcal{A})$.

This graded space has a natural coboundary called the Hochschild differential, which is related to the product in $\mathcal{A}$ and will be denoted δ_m (see e.g. [14, 15] and appendix A for a definition). As was shown in [1], the BRST operator on the closed string operators corresponds to the Hochschild differential δ_m, at least if we assume that the boundary operators are taken on-shell. Otherwise there is a correction from the BRST operator. More precisely, the closed string BRST operator acting on $\hat{\phi}_I$ corresponds to $\delta_m + \delta_Q$ acting on Φ_I, where δ_Q is the supercommutator with the open string BRST operator Q, where the action on several boundary operators is interpreted appropriately. More generally, when other structure constants of the A_∞ algebra are nonzero, we find that the closed string BRST operator corresponds to $\sum_n \delta_{m_n}$, where δ_{m_n} is defined analogously to the Hochschild differential, using the canonical Gerstenhaber bracket $[\cdot,\cdot]$ on the Hochschild complex. More precise definitions of the Gerstenhaber bracket and the Hochschild differential can be found in appendix A.

2.2 Deformations of dg-algebras and Gerstenhaber algebras

In the case of a dg-algebra, that is we only have $Q = m_1$ and an associative product $m = m_2$, we find that the Hochschild complex is a double complex, with differentials δ_Q and δ_m. In the rest of this paper we will assume that the undeformed open string theory has this structure of a dg-algebra. The map from closed string operators ϕ_I to the multilinear operations Φ_I through the correlation functions naturally relates the closed string operators to the Hochschild complex. The closed string BRST operator corresponds to the total Hochschild coboundary $\delta_m + \delta_Q$ on this double complex under this map. It follows that the on-shell closed string algebra, that is the BRST cohomology, is naturally related to the Hochschild cohomology $H^*(\mathrm{Hoch}(\mathcal{A}))$, which is the total cohomology of this double complex. To calculate this total cohomology, we can use a spectral sequence calculation. To calculate the first term in the spectral sequence, one can make use of the Hochschild-Kostant-Rosenberg theorem, or the analytic generalization due to Connes. This expresses the cohomology with respect to δ_m of the Hochschild complex of a polynomial algebra as a polynomial algebra. On the first term of the spectral sequence we now have the coboundary δ_Q (or more precisely the induced coboundary in the Hochschild cohomology).

The corresponding cohomology in turn constitutes the second term in the spectral sequence. In general, the spectral sequence does not have to terminate here. Generally, we still have a remnant of δ_m, if it is not completely compatible with the cohomology of δ_Q, in the sense that we can not take homogeneous representatives which are simultaneously closed under both coboundaries. We will see that this indeed happens for HCS. This gives rise to descent equations of the form $\delta_m\Phi^{(n)} = -\delta_Q\Phi^{(n+1)}$. Solving these descent equations, we see that the total sum $\Phi = \sum_n \Phi^{(n)}$ is indeed closed with respect to the total coboundary, $(\delta_m + \delta_Q)\Phi = 0$.

Closed string operators form naturally a very interesting algebraic structure see e.g. [43]–[48]. Well known is the (OPE) product on the closed string. It is given by the constant term in the OPE. On shell, this product is associative and symmetric. Another structure is the bracket, which is found in terms of the contour integral of one operator around each other. In conformal gauge this is the residue, the coefficient of the $1/z$ pole, of the OPE. It is related to the current algebra of the closed string. On-shell, these two operations satisfy the relations of what is called a Gerstenhaber algebra. This is a graded version of a Poisson algebra, for which the bracket has degree -1.

The Hochschild cohomology of an associative algebra is also known to have the structure of a Gerstenhaber algebra. The symmetric product is defined by the so-called cup product, denoted $\cup$ and defined in appendix A, and the bracket is defined by a graded version of composition of multilinear maps. It turns out that for topological strings the map defined by the mixed correlators intertwines the structure of Gerstenhaber algebra of the closed string and the Hochschild cohomology of the open string algebra. In more concrete terms the product $\hat{\phi}_I \cdot \hat{\phi}_J$ of two closed string operators corresponds to the map $\Phi_I \cup \Phi_J$, while the bracket of a pair of operators $\{\hat{\phi}_I, \hat{\phi}_J\}$ corresponds to the Gerstenhaber bracket of the corresponding maps $[\Phi_I, \Phi_J]$ in the Hochschild complex.

2.3 Open string field theory action and gauge invariant observables

We can make contact between the above construction and open string field theory. We saw that in general the topological open string algebra $\mathcal{A}$ has the structure of an A_∞ algebra. We assumed the undeformed case to be a graded differential associative algebra, given by the BRST operator Q and the product m. This can be understood in terms of an open string field theory. The open string field is expanded in the boundary operators as $A = A^a\hat{\alpha}_a$. We will here reduce only to the degree 1 part of the string field. In the explicit case of the D-brane this corresponds to the 1-form gauge field, and gives the physical part of the field. In principle also other degrees could be included, which are interpreted as ghosts and anti-ghosts. In order to write down the action we need an inner product $\langle\cdot,\cdot\rangle$ on the algebra $\mathcal{A}$, which is provided by the 2-point functions. The algebraic structures Q and m will be cyclic using this inner product. The action can then be written as [2, 26]

$$S_0 = \frac{1}{2}\langle A, QA\rangle + \frac{1}{3}\langle A, m(A,A)\rangle\,. \tag{2.4}$$

Often the inner product is written as a (formal) integral. This action has a gauge invariance by

$$\delta^0_\Lambda A = Q\Lambda + m(A,\Lambda) - m(\Lambda,A)\,, \tag{2.5}$$

where Λ is a degree zero field.

We saw above that we can understand the multilinear operations Φ as deformations of the A_∞ structure constants. In this they will also deform the string field theory action above. We can alternatively understand them as observables in the open string field theory. Indeed observables can be used to deform the action by exponentiating them. An important criterion for these observables is that they respect the gauge invariance. We will now see that indeed, in an appropriate sense, they do.

Let us first consider the case of a closed Φ having only a component $\Phi^{(0)}$ of order 0. This implies that it is closed with respect to both coboundaries, $\delta_Q\Phi^{(0)} = 0 = \delta_m\Phi^{(0)}$. As $\Phi^{(0)} \in C^0(\mathcal{A},\mathcal{A}) = \mathcal{A}$, we can define an observable

$$\mathcal{O}_\Phi = \left\langle A, \Phi^{(0)} \right\rangle. \tag{2.6}$$

This can be heuristically considered as an "integral" of the "gauge field" A over a "cycle" given by $\Phi^{(0)}$. In the analogue of string field theory with noncommutative geometry [2], it is natural to relate the multilinear maps Φ with cycles. We saw they are elements of the Hochschild cohomology, while in noncommutative geometry the latter is related to the cycles in the noncommutative space [10]. To see the gauge invariance of this observable we note the following two consequences of the above restriction on $\Phi^{(0)}$. From $\delta_Q\Phi^{(0)} = 0$ we have

$$\left\langle Q\Lambda, \Phi^{(0)} \right\rangle = -\left\langle \Lambda, \delta_Q\Phi^{(0)} \right\rangle = 0\,. \tag{2.7}$$

The constraint $\delta_m\Phi^{(0)} = 0$ implies

$$\left\langle m(A,\Lambda) - m(\Lambda,A), \Phi^{(0)} \right\rangle = \left\langle \Lambda, \delta_m\Phi^{(0)}(A) \right\rangle = 0\,. \tag{2.8}$$

Combining these, we find that $\mathcal{O}_\Phi$ is indeed gauge invariant.

Let us now look at a more general case, where Φ has components of any order. We still take Φ to be closed; so the components satisfy $\delta_m\Phi^{(n)} + \delta_Q\Phi^{(n+1)} = 0$. Also the maps $\Phi^{(n)}$ can be shown to be cyclic. We take the following ansatz for the corresponding observable

$$\mathcal{O}_\Phi = \sum_{n\geq 0} \frac{(-1)^{\frac{1}{2}(n-1)(n-2)}}{n+1} \left\langle A, \Phi^{(n)}(A,\cdots,A) \right\rangle. \tag{2.9}$$

We can then derive the following expression for the variation of the deformed action under the gauge transformations

$$\delta^0_\Lambda \mathcal{O}_\Phi = \sum_{n\geq 1}\sum_{i=1}^{n} (-1)^{\frac{1}{2}(n-1)(n-2)+i} \left\langle F, \Phi^{(n)}(A,\cdots,A,\underset{i}{\Lambda},A,\cdots,A) \right\rangle. \tag{2.10}$$

where the i indicates that Λ is inserted at the ith place, and $F = QA + m(A,A)$ is the "field strength" of A. As $F = \delta S_0/\delta A \approx 0$ is the equation of motion of the undeformed theory, we find gauge invariance on shell, at least to first order in the deformation. This observation can be used to find full gauge invariance to first order. We modify the gauge transformation law by terms involving Φ,

$$\delta'_\Lambda A = \sum_{n\geq 1}\sum_{i=1}^{n}(-1)^{\frac{1}{2}n(n+1)+i}\Phi^{(n)}(A,\cdots,A,\underset{i}{\Lambda},A,\cdots,A)\,. \tag{2.11}$$

Then the variation of the undeformed action S_0 will exactly cancel the deviation form gauge invariance of $\mathcal{O}_\Phi$ above. Notice that this is quite natural, as $\Phi^{(n)}$ are the deformation of the A_∞ structure. So although the $\mathcal{O}_\Phi$ are now not genuinely gauge invariant expressions, the total expression $S_0+\mathcal{O}_\Phi$ is gauge invariant under the modified gauge transformations $\delta^0_\Lambda+\delta'_\lambda$, to first order in ϕ.

As $\mathcal{O}_\Phi$ transforms nontrivially under the modification δ'_Λ the combination $S_0+\mathcal{O}_\Phi$ is not exactly gauge invariant with respect to the modified gauge transformations. The corrections are however higher order in the deformation ϕ. We can generalize the gauge invariance to higher orders; we should then however also take into account the higher order corrections to the maps Φ. We therefore replace them by the completely deformed multilinear operations $\bar\Phi$. They can be derived from the fully deformed correlation functions, and should satisfy the master equation

$$\delta_Q\bar\Phi^{(n)}+\delta_m\bar\Phi^{(n-1)}+\sum_{k=0}^{n}(-1)^{(n-1)k}\bar\Phi^{(n+1-k)}\circ\bar\Phi^{(k)}=0\,. \tag{2.12}$$

In [1] it was shown that this is satisfied by the mixed correlators completely deformed by a closed string operator ϕ provided it is BRST closed and it satisfies the additional identity $\{\phi,\phi\}=0$.

The gauge deformation is deformed by δ'_Λ given by the same formula (2.11) replacing Φ by $\bar\Phi$, and $\delta'_\Lambda S_0=\langle F,\delta'_\Lambda A\rangle$. The variation of the observable under a gauge transformation can be written as

$$\delta_\Lambda\mathcal{O}_{\bar\Phi}=\sum_n(-1)^{\frac{1}{2}(n-1)(n-2)}\left\langle\delta_\Lambda A,\bar\Phi^{(n)}(A,\cdots,A)\right\rangle. \tag{2.13}$$

Using the cyclicity of the correlation functions $\bar\Phi$, we can derive the following identities

$$\begin{aligned}\left\langle\Lambda,\delta_Q\bar\Phi^{(n)}(A^n)\right\rangle &= -\left\langle Q\Lambda,\bar\Phi^{(n)}(A^n)\right\rangle+\\ &\quad+\sum_k(-1)^{n+k}\left\langle QA,\bar\Phi^{(n)}(A^k,\Lambda,A^{n-k-1})\right\rangle,\\ \left\langle\Lambda,\delta_m\bar\Phi^{(n-1)}(A^n)\right\rangle &= (-1)^{n-1}\left\langle m(A,\Lambda)-m(\Lambda,A),\bar\Phi^{(n-1)}(A^{n-1})\right\rangle+\\ &\quad+\sum_k(-1)^{k+1}\left\langle m(A^2),\bar\Phi^{(n-1)}(A^k,\Lambda,A^{n-k-2})\right\rangle,\\ \left\langle\Lambda,\bar\Phi^{(n+1-l)}\circ\bar\Phi^{(l)}(A^n)\right\rangle &= \sum_k(-1)^{n-k-l}\left\langle\bar\Phi^{(l)}(A^l),\bar\Phi^{(n+1-l)}(A^k,\Lambda,A^{n-k-l})\right\rangle.\end{aligned} \tag{2.14}$$

Here A^n stands for the n times repeated arguments A. The sum of the left hand sides is the master equation, and therefore vanishes by our assumption. Summing over the right hand sides including a proper sign, one finds they sum up to $\delta^0_\Lambda \mathcal{O}_{\bar{\Phi}} + \delta'_\Lambda S_0 + \delta'_\Lambda \mathcal{O}_{\bar{\Phi}}$. This shows that the deformed action $S_0 + \mathcal{O}_{\bar{\Phi}}$ is gauge invariant for the full gauge transformation $\delta^0_\Lambda + \delta'_\Lambda$.

Having identified the closed string operators with the elements of the Hochschild cohomology, we can now invert the construction of gauge invariant operators. That is, for any element Φ of the Hochschild cohomology of the open string algebra $\mathcal{A}$ we construct the above operator S. The descent equation satisfied by the cohomology classes guarantee that this is gauge invariant at least to first order in the deformed gauge invariance.

3. The B-model

We now discuss in some detail the topological open string related to Kodaira-Spencer theory, which is described by holomorphic Chern-Simons [26]. This topological string is also known as the B-model. A more detailed discussion of the B-model can be found in [23].

3.1 Action and BRST symmetry

We consider a Calabi-Yau space M. In general, we could also consider a general complex manifold, but for non-Calabi-Yau spaces there is an anomaly which makes it hard to define a physical theory (the theory is not unitary anymore).

The topological B-model is a twisted version of the $(2,2)$ supersymmetric CFT with target space M. The fields of this topological sigma-model are the bosonic coordinate fields $z^\mu, \bar{z}^{\bar{\mu}}$, two sets of twisted fermions $\bar{\eta}^{\bar{\mu}}, \chi_\mu$ transforming as worldsheet scalars, and a set of twisted fermions ρ^μ transforming as worldsheet 1-forms. The action is given by

$$S = t\int_\Sigma \left(g_{\mu\bar{\mu}} dz^\mu * d\bar{z}^{\bar{\mu}} - g_{\mu\bar{\mu}} \rho^\mu * D\bar{\eta}^{\bar{\mu}} \right) + \frac{1}{\kappa}\int_\Sigma \left(\rho^\mu D\chi_\mu - \frac{1}{2} R^\lambda_{\mu\bar{\mu}\nu} \rho^\mu \rho^\nu \bar{\eta}^{\bar{\mu}} \chi_\lambda \right), \qquad (3.1)$$

where $R^\lambda_{\mu\bar{\mu}\nu} = g^{\lambda\bar{\lambda}} R_{\bar{\lambda}\mu\bar{\mu}\nu}$ is the curvature of the Kähler manifold M and D is a covariantized derivative. In the action we have introduced an extra coupling parameter. Usually the two coupling parameters are identified as $t = 1/\kappa$. For our purpose it will be more useful to leave them as independent parameters.

This action has a BRST symmetry Q given by

$$Q\bar{z}^{\bar{\mu}} = \bar{\eta}^{\bar{\mu}}\,, \qquad Q\rho^\mu = dz^\mu\,, \qquad Qz^\mu = Q\bar{\eta}^{\bar{\mu}} = Q\chi_\mu = 0\,. \qquad (3.2)$$

Identifying the $\bar{\eta}^{\bar{\mu}}$ with the 1-forms $d\bar{z}^{\bar{\mu}}$ on M, we can identify Q with the Dolbeault differential $\bar{\partial}$ on M. The action (3.1) can be written as the sum of a topological term and a BRST exact term, as

$$S = \frac{1}{\kappa}\int_\Sigma \rho^\mu d\chi_\mu + Q\int_\Sigma \left(t g_{\mu\bar{\mu}} \rho^\mu * d\bar{z}^{\bar{\mu}} - \frac{1}{2\kappa} \Gamma^\lambda_{\mu\nu} \rho^\mu \rho^\nu \chi_\lambda \right). \qquad (3.3)$$

As all the t-dependence is in the exact terms, this shows that the B-model is independent of this parameter. Similarly, we see from this that the B-model does not depend on the Kähler class of the metric $g_{\mu\bar{\mu}}$ and on the worldsheet metric. It depends only on the complex structure of M. As t can be identified with an inverse coupling constant, this also means that the lowest order weak coupling expansion is exact. This will turn out very useful in calculating correlation functions, as we can safely take the limit $t \to \infty$. Because the last term in the action is only BRST closed, the B-model potentially does depend on the coupling κ. Although this dependence was somehow suppressed in the pure open string case [26], we will see that this is not necessarily true for the coupling between the open and the closed sector. The dependence will however be rather mild and can mostly be undone by a rescaling. There will be a crucial difference between vanishing and nonvanishing values of κ.

On closed worldsheets there exists a second BRST operator Q' given by

$$Q'\bar{z}^{\bar{\mu}} = \frac{g^{\bar{\mu}\mu}}{\kappa t}\chi_\mu\,, \quad Q'\rho^\mu = *\,dz^\mu\,, \quad Q'z^\mu = Q'\chi_\mu = 0\,, \quad Q'\bar{\eta}^{\bar{\mu}} = -\partial_{\bar{\nu}} g^{\mu\bar{\mu}}\bar{\eta}^{\bar{\nu}}\chi_\mu\,, \tag{3.4}$$

but it will not play a very important role in the present paper. It will be explicitly broken by the boundary conditions, as $\bar{z}^{\bar{\mu}}$ and χ_μ will never have the same boundary condition. It corresponds to the Dolbeault operator $\partial^\dagger$.

It can be shown that indeed the energy momentum tensor is BRST exact, $T_{\alpha\beta} = \{Q, b_{\alpha\beta}\}$, showing that the theory is indeed topological [26]. The 1-form charge G for this current has the following action on the fields

$$\begin{aligned} Gz^\mu &= \rho^\mu\,, \qquad G\bar{z}^{\bar{\mu}} = 0\,, \qquad G\bar{\eta}^{\bar{\mu}} = d\bar{z}^{\bar{\mu}}\,, \\ G\chi_\mu &= \kappa t g_{\mu\bar{\mu}} * d\bar{z}^{\bar{\mu}} - \Gamma^\lambda_{\mu\nu}\rho^\nu\chi_\lambda\,, \qquad G\rho^\mu = -\frac{1}{2}\Gamma^\mu_{\nu\lambda}\rho^\nu\rho^\lambda\,. \end{aligned} \tag{3.5}$$

It can be straightforwardly checked that there is an on-shell relation $\{Q, G\} \approx d$, reflecting the above relation between the currents b and T. This operator can be considered as the target space operators $\bar{\partial}^\dagger$ and ∂. The two BRST operators Q, Q' and the two components G are related by a twist to the the four supersymmetry operators of the $(2,2)$ theory. The operator Q' can be shown to satisfies the analogous relation $\{Q', G\} \approx *\,d$. Therefore we have also $T_{\alpha\beta} = \{Q', b'_{\alpha\beta}\}$, where $b'_{\alpha\beta}$ is related to $b_{\alpha\beta}$ by a Hodge duality on one of the indices — it is the current for $G' = -*\,G$.

3.2 Kodaira-spencer theory

The closed string B-model [23] is governed by the Kodaira-Spencer theory [24], which studies the moduli space of complex structures. KS theory is known quite well; many details from the point of view of the B-model can be found for example in [25].

The operators of the closed B-model are build using the fermionic scalars χ_μ and $\bar{\eta}^{\bar{\mu}}$. The operator corresponding to a form have the form [23]

$$\hat{\phi} = \phi^{\mu_1\ldots\mu_p}_{\bar{\mu}_1\ldots\bar{\mu}_q}(z,\bar{z})\bar{\eta}^{\bar{\mu}_1}\cdots\bar{\eta}^{\bar{\mu}_q}\chi_{\mu_1}\cdots\chi_{\mu_p}\,, \tag{3.6}$$

where $\phi \in \Omega^{0,q}(M, \bigwedge^p \mathcal{T}_M)$ is a $(0,p)$-form with values in the exterior powers of the holomorphic tangent space. We see that the action of the BRST operator Q indeed correspond to the operator $\bar{\partial}$ on these forms. Therefore the BRST closed operators correspond to the closed forms ϕ, and the BRST cohomology of the closed string can be identified with the space of vector valued forms on the complex space M,

$$\bigoplus_{p,q} H^{-p,q}(M) \equiv \bigoplus_{p,q} H^q_{\bar\partial}(M, \textstyle\bigwedge^p \mathcal{T}_M) \cong \bigoplus_{p,q} H^{3-p,q}(M)\,, \tag{3.7}$$

where $\mathcal{T}_M$ denotes the holomorphic tangent space to M, and we take the cohomology with respect to $\bar\partial$. The last equivalence is given by contraction with the holomorphic 3-form Ω.

The physical operators are the closed string operators of total ghost number 2, that is operators for which $p+q=2$. Usually one considers Calabi-Yau's having $h^{1,0}=0$. This implies that the only physical operators are in $H^{-1,1}(M)$. They correspond to the deformation of the complex structure. Here we will also consider more general "CY's" for which the above is not necessarily true, so may have even more reduced holonomy. The 3-point function on the sphere of three physical operators $\hat\phi_I$, $\hat\phi_J$ and $\hat\phi_K$ is given by [23]

$$\left\langle \hat\phi_I \hat\phi_J \hat\phi_K \right\rangle_{S^2} = \int \Omega \wedge \phi_I \cdot \phi_J \cdot \phi_K \cdot \Omega\,. \tag{3.8}$$

Here the holomorphic vector indices are first fully contracted with Ω. Note that there is a selection rule: this is only nonzero when $\sum_I p_I = \sum_I q_I = 3$. This is of course nothing but the conservation of left and right ghost number, which are known to have an anomaly of 3. Also this is satisfied when all 3 are in $H^{-1,1}(M)$. Similarly, when one considers allows for non-physical operators, having different charges, one has obvious n-point functions when the above selection rule is satisfied.

3.3 Holomorphic Chern-Simons theory

The open string sector of the B-model is given by holomorphic Chern-Simons theory (HCS) [26]. The KS theory couples naturally to holomorphic bundles, or more generally, even dimensional D-branes wrapped around holomorphic cycles in the Calabi-Yau with a holomorphic bundle on it. These are the natural boundary conditions which in fact are invariant under the BRST operator of KS. For simplicity, we will mainly talk about bound states containing a nonzero number of D-branes fully wrapped around the Calabi-Yau. The D-brane system is then described by a holomorphic bundle E over the Calabi-Yau. The tangent space of the model, spanned by the boundary operators α_a, are the deformations of the holomorphic connection A of the holomorphic bundle E on the Calabi-Yau. The BRST operator of the open string becomes the covariantized form of the closed string (total) BRST operator, $Q_B = \bar\partial_A$. The derivation condition $Q^2=0$ translates to $F^{0,2} = \bar\partial_A^2 = 0$, expressing the fact that the bundle has to be holomorphic. The tangent space is at any point isomorphic to $\Omega^{0,1}(M, \mathrm{End}(E))$, the space of $(0,1)$-forms with values in the adjoint bundle. We will simply denote the variations by α. The corresponding zero-form vertex operator and its descendant are given by

$$\hat\alpha = \alpha_{\bar\mu}(z,\bar z)\bar\eta^{\bar\mu}\,, \qquad \hat\alpha^{(1)} = \alpha_{\bar\mu}(z,\bar z) d\bar z^{\bar\mu} + \partial_\mu \alpha_{\bar\mu}(z,\bar z)\rho^\mu \bar\eta^{\bar\mu}\,, \tag{3.9}$$

where the σ is a coordinate along the boundary. The fermions χ_μ do not appear in the operators as they satisfy Dirichlet boundary conditions. The BRST cohomology of the open string B-model can therefore be identified with

$$H^*_{\bar{\partial}_A}(M, \mathrm{End}(E))\,. \tag{3.10}$$

Using this correspondence of operators and (variation of the) gauge field, we can express all correlators of HCS in terms of an effective field theory, living on the worldvolume of the D-brane. The effective action is given by [26]

$$S_0 = \int_M \Omega \wedge \mathrm{Tr}\left(\frac{1}{2} A \wedge \bar{\partial} A + \frac{1}{3} A \wedge A \wedge A\right). \tag{3.11}$$

For example, the structure constants of the open string algebra are given in terms of this effective field theory by

$$F_{abc} = \int_M \Omega \wedge \mathrm{Tr}\Big(\alpha_a \wedge \alpha_b \wedge \alpha_c\Big)\,. \tag{3.12}$$

In other words, the theory is given by the cubic OSFT of Witten [2], with BRST operator $\bar{\partial}_A$, product $\wedge$ and integral/trace $\int \Omega \wedge \mathrm{Tr}$.

This discussion can be generalized for B-branes wrapping a holomorphic cycle $C \subset M$. When C is a genuine submanifold of M, there are extra scalars X in the B-model. They can be considered as describing the transverse motion of the brane. They are therefore sections of the holomorphic normal bundle to C, which we will denote $\mathcal{N}_C$. These scalar fields transform in the adjoint of the gauge group. Hence, they will actually give rise to a field $X \in \Omega^0(C, \mathcal{N}_C \otimes \mathrm{End}(E))$. In the following, indices $\mu, \nu, \ldots$ and the corresponding antiholomorphic indices will be used for directions along C, and indices $i, j, \ldots$ for direction in the holomorphic normal bundle $\mathcal{N}_C$. Therefore, the scalar fields will have coordinates X^i. Perturbations of the scalar fields will give rise to operators in the B-model. To distinguish them from the 1-form operators, we will denote them γ. The space of on-shell operators (including the ghost and anti-ghost sectors) corresponds to the cohomology $H^*_{\bar{\partial}_A}(C, \bigwedge^* \mathcal{N}_C \otimes \mathrm{End}(E))$, with the physical operators in degree 0. The boundary operator corresponding to an element $\gamma \in H^0(C, \mathcal{N}_C \otimes \mathrm{End}(E))$ is given by

$$\hat{\gamma} = \gamma^i \chi_i\,, \qquad \hat{\gamma}^{(1)} = \kappa t \gamma^i g_{i\bar{\imath}} * d\bar{z}^{\bar{\imath}} - \gamma^i \Gamma^j_{ki} \rho^k \chi_j + \partial_\mu \gamma^i \rho^\mu \chi_i\,. \tag{3.13}$$

In order for this to be nontrivial on the boundary, the fields χ_i should satisfy Neumann boundary conditions. Furthermore, the fields z^i, $\bar{z}^{\bar{\imath}}$, and $\bar{\eta}^{\bar{\imath}}$ will satisfy Dirichlet boundary conditions.

Let us first assume that C is a point. In this case there are three zero-modes for the fermions χ_i which have to be provided by as many scalar fields. More scalar fields would lead to contractions which vanish when we take $t \to \infty$. Therefore, the complete contribution comes from the zero modes, giving rise to a cubic interaction. The action is

$$S_0 = \frac{1}{3} \Omega \cdot \mathrm{Tr}(XXX)\,, \tag{3.14}$$

where the holomorphic 3-form Ω is evaluated at the point C and fully contracted with the vector indices of the field X. Next consider the case where C is a complex curve in

M. Then there are two zero-modes for the normal fermions χ_i and a single one for $\bar{\eta}^{\bar{\mu}}$. Reducing to zero-mode integrals, again noting that contractions are subleading, we find the action

$$S_0 = \int_C \Omega \cdot \mathrm{Tr}\left(\frac{1}{2} X \bar{\partial} X + XAX\right). \tag{3.15}$$

The first term is induced by the BRST operator. When C a complex surface, there is one zero-mode for χ_i and two for $\bar{\eta}^{\bar{\mu}}$. The action is then

$$S_0 = \int_C \Omega \cdot \mathrm{Tr}(X \bar{\partial} A + XA \wedge A). \tag{3.16}$$

Note that in all three cases the action can be found from dimensional reduction.

4. Mixed correlators in the open-closed B-model

We now turn to our main objective, which is the calculation of correlators in the open-closed B-model.

4.1 Mixed correlators

We will calculate the mixed correlators, involving a single closed string operator, for a brane wrapped completely around M. Therefore, there are only open string operators $\hat{\alpha}$, corresponding to the gauge field.

To calculate the mixed correlators we need the propagators between the various $\beta_I \in H^{0,2}(M)$. There are no possible contractions in the $t \to \infty$ limit, and the correlators is completely determined by the zero-mode integral. As there are three zero-modes for $\bar{\eta}^{\bar{\mu}}$ we need a single boundary insertion, giving a mixed 2-point function

$$\left\langle \hat{\beta}_I \hat{\alpha}_a \right\rangle = \int \beta_I \wedge \Omega \wedge \mathrm{Tr}(\alpha_a). \tag{4.1}$$

Next we consider a closed string operator corresponding to $\varphi \in H^{-1,1}(M)$. As this contains a χ_μ, which has no zero-modes, the 2-point function vanishes. The relevant correlator is a mixed 3-point function $\left\langle \hat{\varphi}_I \hat{\alpha}_a \int \hat{\alpha}_b^{(1)} \right\rangle$. Writing out the operators we have

$$\int d\sigma \left\langle (\varphi_I{}^{\mu}_{\bar{\mu}} \bar{\eta}^{\bar{\mu}} \chi_\mu)(u) (\alpha_{a\bar{\lambda}} \bar{\eta}^{\bar{\lambda}})(0) (\alpha_{b\bar{\nu}} \partial_\sigma \bar{z}^{\bar{\nu}} + \partial_\nu \alpha_{b\bar{\nu}} \rho^\nu_\sigma \bar{\eta}^{\bar{\nu}})(\sigma) \right\rangle. \tag{4.2}$$

Because χ_μ has no zero-modes, the only potential contributions come from contractions of this field. Therefore we need a fermion ρ^ν somewhere in the expression. This implies that we need a first descendant on the boundary, for which we need at least two open string operators. The contraction of χ_μ in $\hat{\varphi}_I$ with ρ^μ in $\hat{\alpha}_b$ will give a propagator $\sim \frac{\kappa}{\sigma - u}$. When we introduce more boundary operators in the correlator (as first descendants), they will not contribute as ρ and $\partial_\sigma \bar{z}$ have no zero-mode and further contractions always give $1/t$ contributions, which vanish for $t \to \infty$. We conclude that the correlator is given by

$$\left\langle \hat{\varphi}_I \hat{\alpha}_a \int \hat{\alpha}_b^{(1)} \right\rangle = \kappa \int \varphi_I \cdot \Omega \wedge \mathrm{Tr}(\alpha_a \wedge \partial \alpha_b). \tag{4.3}$$

We should note at this point that there are potential contributions from the boundary of the integration, when two open string operators collide. The contribution will however always involve a $\bar{\eta}\rho$ or $z\bar{z}$ contraction, and therefore give contributions of order $1/t$, which would vanish in our limit. Though the integrals are potentially divergent, we will regularize them using a point-splitting procedure, thereby avoiding any of these contributions. We will come back to this point in more detail later.

For $\theta \in H^{-2,0}(M)$ the result is

$$\left\langle \hat{\theta}_I \hat{\alpha}_a \int \hat{\alpha}_b^{(1)} \int \hat{\alpha}_c^{(1)} \right\rangle = \frac{\kappa^2}{2} \int \theta_I \cdot \Omega \wedge \mathrm{Tr}(\alpha_a \wedge \partial\alpha_b \wedge \partial\alpha_c) \,. \tag{4.4}$$

The calculation is similar to the one above. Only now $\hat{\theta}_I$ contains two fermions χ_μ which we need to contract. Therefore, we need two descendants, and therefore at least three boundary operators. The contractions of the two χ_μ factors in $\hat{\theta}_I$ with the ρ^μ in the two descendant operators give a nonzero integral proportional to the residue of the OPE. The integral over the insertions of $\hat{\alpha}_b$ and $\hat{\alpha}_c$ can be written as angular integrals, where we have take into account of the order. This will give an integral over a 2-simplex, which is responsible for the factor of $1/2$ in front of the result. We note that in general when there are n first descendants integrated in fixed order along the boundary, this will be an integral over an n-simplex, giving a factor of $1/n!$. More insertions of boundary operators will again give zero, as more contractions give subleading corrections in $1/t$, and therefore these are the only nonzero correlators involving a single $\hat{\theta}_I$.

4.2 Including scalars

Let us now generalize to the case where the B-brane wraps a holomorphic cycle $C \subset M$. We will not discuss this in all detail, but rather look a few examples. With what we have learned so far, this will allow us to find the general rule to construct the interactions. We take for C a complex curve in M, and denote the single complex coordinate z^μ by z. First, we take a closed string operator corresponding to a complex structure deformation of the form $\hat{\phi}_I = \varphi_I{}^i_{\bar{z}} \bar{\eta}^{\bar{z}} \chi_i$, and a single open string operator. We can reduce completely to zero-mode integrals, giving

$$\left\langle \hat{\phi}_I \hat{\gamma}_a \right\rangle = \left\langle (\varphi_I{}^i_{\bar{z}} \bar{\eta}^{\bar{z}} \chi_i)(\gamma_a^j \chi_j) \right\rangle = \int_C dz \wedge d\bar{z}\, \varphi_I{}^i_{\bar{z}} \Omega_{zij}\, \mathrm{Tr}(\gamma_a^j) \,. \tag{4.5}$$

For more open string insertions, we insert descendants of the boundary operators. The descendants contain two terms. Let us consider the mixed 3-point function $\langle \hat{\phi}_I \hat{\gamma}_a \int \hat{\gamma}_b^{(1)} \rangle$. The main term contributing will be

$$\kappa t \left\langle (\varphi_I{}^i_{\bar{z}} \bar{\eta}^{\bar{z}} \chi_z)(\gamma_a^j \chi_j)(\gamma_b^k g_{k\bar{\imath}} \partial_n \bar{z}^{\bar{\imath}}) \right\rangle = \kappa \int_C dz \wedge d\bar{z}\, \partial_k \varphi_I{}^i_{\bar{z}} \Omega_{zij}\, \mathrm{Tr}(\gamma_a^j \gamma_b^k) \,. \tag{4.6}$$

Here we used a contraction between z^k and $\partial_n \bar{z}^{\bar{\imath}}$. Notice that this would naively vanish in the $t \to \infty$ limit as the propagator is of order $1/t$. However this is precisely canceled by the explicit factor of t in front of the correlator (coming from the descent procedure). Furthermore the pure zero-mode contribution vanishes, which makes the $t \to \infty$ limit defined. The other term of the descendant will not contribute as there are no zero-modes or contractions possible. There are higher point functions involving extra insertion of scalar operators, which give similar contributions. I.e, any further operator $\hat{\gamma}_c$ in the correlator gives an insertion of $\gamma_c^l \partial_l$, where the derivative acts on φ.

Next, we take a complex structure deformation of the form $\hat{\phi}_I = \varphi_I{}^z_{\bar{z}} \bar{\eta}^{\bar{z}} \chi_z$. As this does not contain any χ_i, we need at least two open string fields to soak up the corresponding zero-modes. One of these is a descendant. The χ_z has to contract with a ρ^z in the descendant operator. This gives

$$\left\langle \hat{\phi}_I \hat{\gamma}_a \int \hat{\gamma}_b^{(1)} \right\rangle = \kappa \left\langle (\varphi_{I\bar{z}} \bar{\eta}^{\bar{z}} \chi_z)(\gamma_a^i \chi_i)(\partial_z \gamma_b^j \rho^z \chi_j) \right\rangle = \kappa \int_C dz \wedge d\bar{z}\, \varphi_I{}^z_{\bar{z}} \Omega_{zij} \operatorname{Tr}(\gamma_a^i \partial_z \gamma_b^j)\,. \tag{4.7}$$

There are again higher point functions with extra boundary insertions, following the pattern as above.

Note that although we have derivatives of sections of $\mathcal{N}_C$ we do not find any term involving the connection (which is part of the Christoffel connection Γ^i_{zj} of M). However the total expression is still manifestly covariant. This can be seen by writing the relevant contributions in terms of a holomorphic version of the Lie derivative $\mathcal{L}$,

$$(\mathcal{L}_\gamma \varphi)^i = \gamma^j \partial_j \varphi^i - \varphi^\mu \partial_\mu \gamma^i\,, \qquad (\mathcal{L}_\gamma \varphi)^\mu = \gamma^i \partial_i \varphi^\mu\,, \tag{4.8}$$

which indeed is manifestly covariant — if we would have written covariant derivatives the Christoffel connection cancels. The absence of the Christoffel connection is necessary for the model to be independent of the Kähler structure.

The general correlation function for $\varphi_I \in H^{-1,1}(M)$ and with $n+1$ scalar insertions can then be written

$$\left\langle \hat{\phi}_I \hat{\gamma}_{a_0} \int \hat{\gamma}_{a_1}^{(1)} \cdots \int \hat{\gamma}_{a_n}^{(1)} \right\rangle = \kappa^n \int_C dz \wedge d\bar{z}\, \Omega_{zij} \operatorname{Tr}\Big(\gamma_{a_0}^j (\mathcal{L}_{\gamma_{a_1}} \cdots \mathcal{L}_{\gamma_{a_n}} \varphi_I)^i_{\bar{z}}\Big)\,. \tag{4.9}$$

The worldsheet field $\bar{\eta}^{\bar{i}}$ has no zero-modes. Also, it does not give rise to any contraction. Therefore, all closed string operators containing this field will give vanishing correlators. This applies to the other components of $\varphi \in H^{-1,1}(M)$. Also, $\beta \in H^{0,2}(M)$ will not contribute to any correlation function.

A similar analysis can be done for $\theta_I \in H^{-2,0}(M)$. As they do not contain $\bar{\eta}^{\bar{\mu}}$ we need at least one operator corresponding to the gauge field to soak up its zero mode. For $\frac{1}{2}\theta^{ij}\chi_i\chi_j$, the lowest order correlator is a 2-point function given by

$$\left\langle \hat{\phi}_I \hat{\alpha}_a \right\rangle = \frac{1}{2} \int_C \theta_I^{ij} \Omega_{zij} \operatorname{Tr}\Big(\alpha_{a\bar{z}}\Big)\,. \tag{4.10}$$

For the component $\frac{1}{2}\theta^{zj}\chi_z\chi_j$ we need to contract the field χ_z, therefore we need another descendant. This leads to a 3-point function

$$\left\langle \hat{\phi}_I \hat{\alpha}_a \int \hat{\gamma}_b^{(1)} \right\rangle = \kappa \int_C \theta_I^{zi} \Omega_{zij} \operatorname{Tr}(\alpha_{a\bar{z}} \partial_z \gamma_b^j)\,. \tag{4.11}$$

There are higher correlators, which involve derivatives of θ_I.

The general correlation function for $\hat{\phi}_I$ derived from $\theta_I \in H^{-1,1}(M)$ and with one 1-form and n scalar insertions can then be written

$$\left\langle \hat{\phi}_I \hat{\alpha}_{a_0} \int \hat{\gamma}^{(1)}_{a_1} \cdots \int \hat{\gamma}^{(1)}_{a_n} \right\rangle = \frac{\kappa^n}{2} \int_C dz \wedge d\bar{z}\, \Omega_{zij}\, \mathrm{Tr}\Big(\alpha_{a_0 \bar{z}} (\mathcal{L}_{\gamma_{a_1}} \cdots \mathcal{L}_{\gamma_{a_n}} \theta_I)^{ij}\Big). \tag{4.12}$$

4.3 Background gauge fields

We will now generalize the calculation of the mixed correlators to the presence of a background gauge field. We will only consider the pure gauge field case. The gauge field couples to the string by a boundary term

$$S_A = \int_{\partial\Sigma} d\sigma\, A^{(1)} = \int_{\partial\Sigma} d\sigma \Big(A_{\bar{\mu}} \partial_\sigma \bar{z}^{\bar{\mu}} - i F_{\mu\bar{\mu}} \rho^\mu \bar{\eta}^{\bar{\mu}} \Big). \tag{4.13}$$

Equivalently, the effect of turning on a background gauge field can be accomplished by adding the exponentiated operator $\exp(\int \hat{A}^{(1)})$ in the correlators. This will add extra terms as there are now also contractions to the extra operators $\hat{A}^{(1)}$. Of course these extra contributions can also be understood in terms of Feynman diagrams built on vertices found above, by contracting some external lines to the background gauge field. For $\varphi \in H^{-1,1}(M)$ the extra correlators are given by

$$\langle \hat{\varphi} \hat{\alpha}_a \rangle = \kappa \int \varphi \cdot \Omega \wedge \mathrm{Tr}(F^{1,1} \wedge \alpha_a). \tag{4.14}$$

Similarly, for $\theta \in H^{-2,0}(M)$ there are mixed correlators

$$\left\langle \hat{\theta} \hat{\alpha}_a \right\rangle = \frac{\kappa^2}{2} \int \theta \cdot \Omega \wedge \mathrm{Tr}(F^{1,1} \wedge F^{1,1} \wedge \alpha_a), \tag{4.15}$$

$$\left\langle \hat{\theta} \hat{\alpha}_a \int \hat{\alpha}^{(1)}_b \right\rangle = \frac{\kappa^2}{2} \int \theta \cdot \Omega \wedge \mathrm{Tr}(\alpha_a \wedge F^{1,1} \wedge \partial \alpha_b + \alpha_a \wedge \partial \alpha_b \wedge F^{1,1}). \tag{4.16}$$

All the different vertices are summarized in figure 1. We used a wiggle for the closed string propagators and a line for the open string propagators. The coupling to the background is indicated by a dashed line.

A similar analysis can be performed for the scalars, by introducing a boundary term $S_X = \int_{\partial\Sigma} d\sigma\, X^{(1)}$. They can be found using the Feynman diagrams, introducing coupling to the background.

4.4 Regularization and effective field theory

We now shortly discuss an important point about regularization, which will explain the difference between the effective and fundamental string field theory.

In calculating the correlation functions, we have to be careful about the boundaries of integration, where two boundary operators approach each other. We consider the collision of two boundary operators inside a correlation function of the form

$$\int d\sigma \left\langle \cdots \hat{\alpha}^{(1)}_a(\sigma) \hat{\alpha}_b(0) \cdots \right\rangle. \tag{4.17}$$

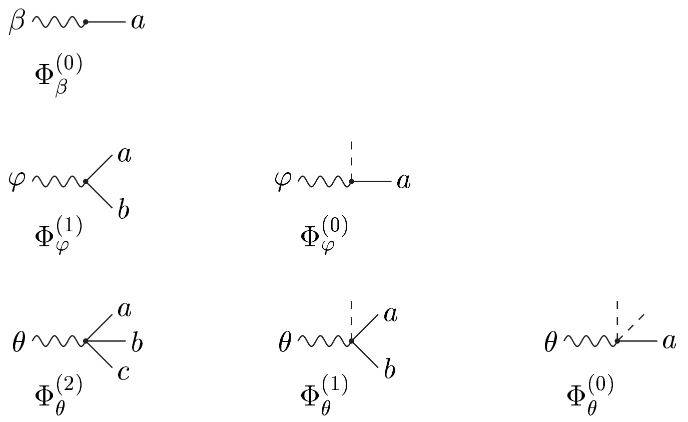

Figure 1: The mixed vertices of the fundamental open-closed string field theory. The dashed lines denote the coupling to the background gauge field. Below the diagrams is the corresponding map $\Phi_I^{(n)}$.

Because we are considering a topological string theory, the contribution will come from the collision point itself, here the boundary of integration at 0. This can be seen by introducing an auxiliary metric of magnitude t on the worldsheet, and taking $t \to \infty$. As the theory is topological, the correlators will not depend on t. In the $t \to \infty$ limit the correlation will be naively zero, as the propagator is proportional to $1/t$. The only contribution can therefore come from the diverging collision. Therefore we study the contribution form this collision, and take an upper limit for the integral of λ. Furthermore we will also cut off the integral at a lower bound of ϵ. We will assume that contraction of the operators will have a singularity in the OPE, which is of the form $1/t\sigma$, where t is the magnitude of the worldsheet metric. As we are in a topological limit [26]

$$\frac{1}{t}\int_\epsilon^\lambda \frac{d\sigma}{\sigma}\,\sigma^{\Delta/t} = \frac{1}{\Delta}\left(\lambda^{\Delta/t} - \epsilon^{\Delta/t}\right). \tag{4.18}$$

When we take $\epsilon \to 0$ first, then the $t \to \infty$ limit would produce an insertion of the propagator $1/\Delta$. However, when we adopt a point splitting procedure by taking $\epsilon \to 0$ only at the end of the calculation, we first take the $t \to \infty$ limit, and this will give a vanishing contribution.

In the calculations above we used the point splitting regularization, $\epsilon > 0$. Let us now see what we get when we take the alternative regularization with $\epsilon = 0$. To discern from the correlators in the point splitting regularization we will denote the correlators and maps by $\tilde{\Phi}_{Ia_0\ldots a_n}$ and $\tilde{\Phi}_I^{(n)}$, respectively.

Let us calculate the extra contributions to the mixed 3-point function

$$\tilde{\Phi}_{Iab} = \left\langle \hat{\phi}_I \hat{\alpha}_a \int \hat{\alpha}_b^{(1)} \right\rangle, \tag{4.19}$$

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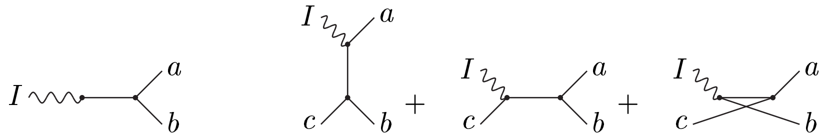

Figure 2: The Feynman diagram contributing to the effective mixed 3-point function $\tilde{\Phi}_{Iab}$ and the three Feynman diagrams contributing to the mixed 4-point function $\tilde{\Phi}_{Iabc}$.

coming from the collision of boundary operators. First, we have to insert the first descendants. We get

$$\tilde{\Phi}_{Iab} = \Phi_{Iab} + \int d\sigma \, \langle (\varphi_I \bar{\eta} \chi)(u)(\alpha_a \bar{\eta})(0)(\alpha_b \partial_\sigma \bar{z} + \partial \alpha_b \rho \bar{\eta})(\sigma) \rangle \, . \tag{4.20}$$

There are obvious contractions of indices, which we only indicated by the brackets. The contraction of ρ with the closed string operator was responsible for the correlation function (4.3); here we are interested in the other contributions. Using that we can calculate this in the limit $t \to \infty$, these can only come from the poles in the integral where vertex operators collide, at the boundaries of the integration. There are $z\partial_\sigma \bar{z}$ and $\bar{\eta}\rho$ contractions, which can be summarized as

$$\tilde{\Phi}_{Iab} = \Phi_{Iab} + \frac{1}{t} \int d\sigma \, \frac{1}{\sigma} \left\langle (\varphi_I \bar{\eta} \chi)(u) \sigma^{\Delta/t} \bar{\eta} \bar{\partial}^\dagger (\alpha_a \alpha_b)(0) \right\rangle , \tag{4.21}$$

The field χ_μ in the closed string operator, having no zero-modes, should contract to a background gauge field. This also provides the extra $\bar{\eta}$ needed to soak up the third zero-mode. Replacing the correlators with zero-mode integrals, we find in the $t \to \infty$ limit

$$\tilde{\Phi}_{Iab} = \Phi_{Iab} + \kappa \int \varphi_I \cdot \Omega \wedge \mathrm{Tr}\left(F^{1,1} \wedge \frac{\bar{\partial}_A^\dagger}{\Delta_A} (\alpha_a \wedge \alpha_b) \right) . \tag{4.22}$$

This expression has a nice interpretation in terms of Feynman diagrams. Noting that $\bar{\partial}_A^\dagger / \Delta_A$ can be identified with the propagator $\bar{\partial}^{-1}$ (after a gauge fixing), this expression can be identified with the first Feynman diagram depicted in figure 2. Here we used the vertices derived earlier in a background gauge field, not explicitly depicting the coupling to the background.

The mixed four-point function

$$\tilde{\Phi}_{Iabc} = \left\langle \hat{\phi}_I \hat{\alpha}_a \int \hat{\alpha}_b^{(1)} \int \hat{\alpha}_c^{(1)} \right\rangle \tag{4.23}$$

can be calculated in a similar way. Now because there are two integrations, we find two contractions; between every adjacent pair. Remember that the boundary of a disc is a circle, so there is also a collision between α_a and α_b. The result is

$$\tilde{\Phi}_{Iabc} = \kappa \int \varphi_I \cdot \Omega \wedge \mathrm{Tr}\left(\partial \alpha_a \wedge \frac{\bar{\partial}_A^\dagger}{\Delta_A} (\alpha_b \wedge \alpha_c) \right) \pm 2 \text{ perms} \, . \tag{4.24}$$

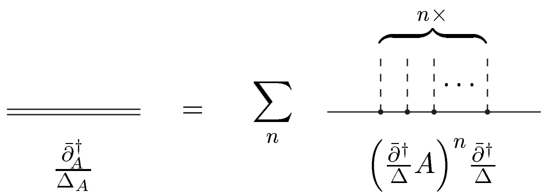

Figure 3: Dyson series for the propagator in a background gauge field, leading to the covariant propagator.

The three cyclic permutations in this formula can be interpreted in terms of the three tree level Feynman diagrams for this correlator, as depicted in figure 2.

In general, the collision between adjacent operators generate all tree level Feynman diagrams. We conclude that this regularization produces the effective field theory. In the background gauge field the BRST operator will also change to the covariant operator $\bar\partial_A$. This can of course be understood as usual by a shift of the open string field.

In the effective theory, there are corrections from collisions between open string operators and the background gauge field. The propagator $\bar\partial_A^{-1} = \bar\partial^\dagger/\Delta$ gets contributions from a Dyson series involving tadpoles to the background. This leads to the covariant form of the propagator $\bar\partial_A^{-1} = \bar\partial_A^\dagger/\Delta_A$, as depicted in figure 3.

The effective theory is relevant for the superpotential of the dimensionally reduced theory. This has been studied recently in the context of the open string B-model in [11, 50, 51].

5. Closed strings as deformations

5.1 Deformations

A complex structure deformation $\varphi \in H^{-1,1}(M)$ acts on the covariant antiholomorphic derivative as

$$\delta\bar\partial_A = \kappa\varphi\cdot\partial\,, \tag{5.1}$$

as can be seen from (4.3). This should be compared to the formula for the variation of the BRST operator in closed string theory due to a deformation by an operator $\hat\phi$, which is given by [52]

$$\delta Q = \oint \hat\phi^{(1)}\,. \tag{5.2}$$

We observe that the operator $\kappa\varphi\cdot\partial$ has the interpretation of the integral of the action of the first descendant of the corresponding bulk operator.

Next, we consider the deformations (4.2). They correspond to a deformation of the cubic term in the effective action, and therefore a deformation of the product. Indeed, the first order deformation by an element $\theta \in H^{-2,0}(M)$ can be written

$$\frac{\kappa^2}{2}\int \Omega\wedge \mathrm{Tr}(\alpha_a\wedge\theta^{ij}\partial_i\alpha_b\wedge\partial_j\alpha_c)\,. \tag{5.3}$$

This is precisely the first order deformation for the noncommutative star-product corresponding to the bivector θ,

$$\alpha_a \star_\theta \alpha_b = \alpha_a \wedge \alpha_b + \frac{\kappa^2}{2}\theta^{ij}\partial_i\alpha_a \wedge \partial_j\alpha_b + \mathcal{O}(\kappa^4\theta^2) \tag{5.4}$$

This was expected, as this deformation corresponds (among others) to a B-field, which we know induces this star-product. In fact, we will argue later that higher order correlators in θ will generate the full star product, given by Kontsevich's formula for deformation quantization.

The last deformation (4.1) corresponds to a shift of the field-strength $F^{0,2}$ by the corresponding element of $H^{0,2}(M)$. Note that here and in the deformation above we probably used a different regularization than usually in the context of noncommutative gauge theories, as here there is an explicit shift of the field strength, as in the case of the commutative description, while in the former case the full $(-2,0)$-form contributes to the star-product, and not only the real part (corresponding to the B-field; the imaginary part corresponds to the Kähler form).

5.2 Descent equations

In calculating the correlation functions, we saw that apart from the leading correlation functions, there were also correlation functions with less open string insertions. In these correlation functions, the $(1,1)$-part of the field strength $F^{1,1}$ plays an important role. Here we explain the interpretation of these lower order correlation functions and how $F^{1,1}$ comes in.

To understand it somewhat better, we interpret the correlation functions again in terms of multilinear maps on the open string algebra, as in (2.3). We start from the correlation functions involving a closed string operator associated with $\varphi \in H^{-1,1}(M)$, related to a complex structure deformation. For such an operator, the leading component with the highest number of open string operators was given in (4.3) of open string operators had two boundary operator. Therefore it corresponds to a linear map, and write this correlator as $\langle\hat{\alpha}_a\Phi^{(1)}_\varphi(\hat{\alpha}_b)\rangle$. The next highest order map represented by the mixed 2-point function will be denoted as $\langle\hat{\alpha}_a\Phi^{(0)}_\varphi\rangle$. We see from the explicit formulas of the correlation functions that

$$\Phi^{(1)}_\varphi(\alpha) = \varphi\cdot\partial\alpha\,, \qquad \Phi^{(0)}_\varphi = \varphi\cdot F^{1,1}\,. \tag{5.5}$$

To understand the relation between these components, and especially the way the field strength arises, consider the following identity,

$$\{\bar\partial_A, \varphi\cdot\partial\} = -\varphi\cdot F^{1,1}\,. \tag{5.6}$$

This equation should be read as an equation for operators acting on adjoint forms $\alpha \in \Omega^{0,*}_{\bar\partial_A}(M, \mathrm{End}(M))$,

$$\bar\partial_A(\varphi\cdot\partial\alpha) + \varphi\cdot\partial(\bar\partial_A\alpha) = -(\varphi\cdot F^{1,1})\wedge\alpha + \alpha\wedge(\varphi\cdot F^{1,1}\,). \tag{5.7}$$

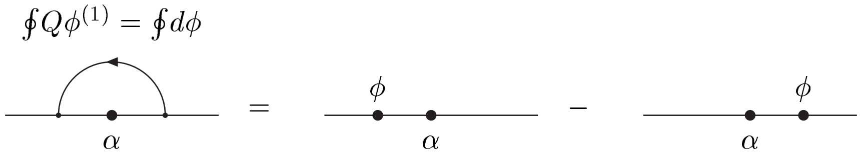

Figure 4: The relation between the descent equation in the bulk, and the action by a commutator at the boundary.

This relation is directly related to the descent equation in the closed string, which reads $\{Q, \phi^{(1)}\} = d\phi$. In the open string B-model, the operator $\bar\partial_A$ was the BRST operator. Notice that integrating $d\phi$ on a half circle around a boundary operator can be written, using Stokes, as two boundary terms, of ϕ moved to the boundary to either side of the boundary operator. This then corresponds to a commutator with the boundary operator induced by ϕ, see figure 4. On the other hand, integrating $\phi^{(1)}$ over a half-circle produces the action $\Phi^{(1)}$ on the boundary operator. From this, we learn the effect of a boundary operator induced by a bulk operator: the bulk operator corresponding to φ acts by commutation with $\varphi \cdot F^{1,1}$. In fact, we can only read off the action by a commutator, and not the (star) product. The latter one can of course easily be guessed to be just the corresponding action with just the wedge product. For most applications, this will not be needed however.

This relation also reflects the descent equation in Hochschild cohomology, which were derived in [1] from a Ward identity. The anticommutator with the BRST operator $\{Q, \Phi^{(1)}_\varphi\}$ in the left-hand side of (5.6) is precisely the action of the coboundary δ_Q in the Hochschild double complex. The right-hand side can actually be interpreted in terms of the usual Hochschild coboundary, related to the product, $\delta_m \Phi^{(0)}_\varphi$. Hence we can write this equation in the form

$$\delta_m \Phi^{(0)}_\varphi = -\delta_Q \Phi^{(1)}_\varphi \,. \tag{5.8}$$

This is the descent equation in the Hochschild double complex with the two coboundaries δ_Q and δ_m. Notice that there is no lower descendant, as $\delta_Q \Phi^{(0)}_\varphi = 0$ due to the Bianchi identity and $F^{0,2} = \bar\partial_A^2 = 0$. Also note that the top component satisfies $\delta_m \Phi^{(1)}_\varphi = 0$, which is the derivation condition of $\varphi \cdot \partial$.

The relation between the descent equation can also be performed for the other closed string operators. For $\theta \in H^{-2,0}(M)$ we can write equation (4.2) in terms of a map $\Phi^{(2)}_\theta$ as $\langle \hat\alpha_a \Phi^{(2)}_\theta(\hat\alpha_b, \hat\alpha_c)\rangle$. The following equation we write as $\langle \hat\alpha_a \Phi^{(1)}_\theta(\hat\alpha_b)\rangle$, and the next as $\langle \hat\alpha_a \Phi^{(0)}_\theta\rangle$. From the explicit expressions for the correlators we have

$$\Phi^{(2)}_\theta(\alpha_a, \alpha_b) = \frac{1}{2}\theta\cdot(\partial\alpha_a \wedge \partial\alpha_b)\,, \qquad \Phi^{(1)}_\theta(\alpha_a) = \theta\cdot(F^{1,1}\wedge\partial\alpha_a)\,, \qquad \Phi^{(0)}_\theta = \frac{1}{2}\theta\cdot(F^{1,1}\wedge F^{1,1})\,. \tag{5.9}$$

Using the above relation, one easily sees that the latter two maps are indeed descendants, that is $\delta_m \Phi^{(1)}_\theta = -\delta_Q \Phi_\theta$ and $\delta_m \Phi^{(2)}_\theta = -\delta_Q \Phi^{(1)}_\theta$. Also, indeed Φ_θ can be considered a deformation of the wedge-product.

For the element $\beta \in H^{0,2}(M)$ we write (4.1) as $\langle \hat\alpha_a \Phi_\beta\rangle$. That is, we simply have $\Phi_\beta = \beta$. There are no other descendants, as $\delta_Q \Phi_\beta = \bar\partial\beta = 0$ and $\delta_m \Phi_\beta = [\beta, \cdot] = 0$.

$(-p,q)$	$\hat{\phi}_I$	$\Phi_I^{(0)}$	$\Phi_I^{(1)}$	$\Phi_I^{(2)}$
$(-2,0)$	$\frac{1}{2}\theta^{\mu\nu}\chi_\mu\chi_\nu$	$\frac{1}{2}\theta^{\mu\nu}F_{\mu\bar{\mu}}F_{\nu\bar{\nu}}\bar{\eta}^{\bar{\mu}}\bar{\eta}^{\bar{\nu}}$	$\theta^{\mu\nu}F_{\mu\bar{\mu}}\bar{\eta}^{\bar{\nu}}\partial_\nu$	$\frac{1}{2}\theta^{\mu\nu}\partial_\mu\wedge\partial_\nu$
$(-1,1)$	$\varphi^\mu_{\bar{\mu}}\bar{\eta}^{\bar{\mu}}\chi_\mu$	$\varphi^\mu_{\bar{\mu}}F_{\mu\bar{\nu}}\bar{\eta}^{\bar{\nu}}\bar{\eta}^{\bar{\nu}}$	$\varphi^\mu_{\bar{\mu}}\bar{\eta}^{\bar{\mu}}\partial_\mu$	
$(0,2)$	$\frac{1}{2}\beta_{\bar{\mu}\bar{\nu}}\bar{\eta}^{\bar{\mu}}\bar{\eta}^{\bar{\nu}}$	$\frac{1}{2}\beta_{\bar{\mu}\bar{\nu}}\bar{\eta}^{\bar{\mu}}\bar{\eta}^{\bar{\nu}}$		

Table 1: Correspondence of elements in the cohomology to elements of the Hochschild complex, for $p+q=2$.

5.3 Relation to the Hochschild complex

The closed string operators correspond to multilinear maps on the open string algebra $\mathcal{A} = \Omega^{0,*}(M, \mathrm{End}(E))$, which satisfy descend equation for on-shell closed string algebras as we saw above. This just says that they correspond to elements of the Hochschild cohomology. Therefore, we find a direct correspondence between the closed string BRST cohomology which was identified with $H^{-*,*}(M)$ and the Hochschild cohomology $H(\mathcal{A})$. Using this, we can actually identify them, as we will see in more detail later. On the other hand, the correlation functions give explicit elements of the Hochschild complex $C^*(\mathcal{A},\mathcal{A})$ of multilinear maps. This can be seen as a formality map for the corresponding Hochschild complex. Here, we will give the clarify this relation by relating the element in cohomology with the leading component in the Hochschild cohomology.

The number of elements in the closed string cohomology $H^{-*,*}(M)$ can be summarized in terms of the Hodge diamond of the Calabi-Yau manifold. This Hodge diamond, in the slightly unusual notation of the more natural cohomology, is given by

$$\begin{array}{ccccccc} & & & h^{-3,0} & & & \\ & & h^{-2,0} & & h^{-3,1} & & \\ & h^{-1,0} & & h^{-2,1} & & h^{-3,2} & \\ h^{0,0} & & h^{-1,1} & & h^{-2,2} & & h^{-3,3} \\ & h^{0,1} & & h^{-1,2} & & h^{-2,3} & \\ & & h^{0,2} & & h^{-1,3} & & \\ & & & h^{0,3} & & & \end{array} = \begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \tag{5.10}$$

where $h^{-p,q} = \dim H^{-p,q}(M) = h^{3-p,q}$.

Any element ϕ_I of the closed string cohomology corresponds through the correlation functions to a sequence of multilinear maps $\Phi_I^{(n)}$. These maps correspond to elements of the Hochschild cohomology of the open string algebra. As we saw, the sequence of maps satisfy the descent equations $\delta_m\Phi_I^{(n)} = -\delta_Q\Phi_I^{(n+1)}$. We assume that these will terminate at a certain maximum value of n for any given ϕ_I. Indeed, this was true in the cases we studied. We will concentrate on this leading component with the largest order. Due to the relation between the closed string cohomology, the Dolbeault cohomology, and the Hochschild cohomology there should be a natural correspondence between the Hodge diamond and the Hochschild cohomology. To motivate this relation, let us look at what we found. For $\phi_I \in H^{-p,q}(M)$ with $p+q=2$ we already saw the relation between the element in cohomology and the leading order map. They are summarized in table 1. These corre-

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$$
\begin{array}{ccccccc}
 & & & \mathrm{Hom}(\mathcal{A}^{\otimes 3},\mathcal{A})^0 & & & \\
 & & \mathrm{Hom}(\mathcal{A}^{\otimes 2},\mathcal{A})^0 & & \mathrm{Hom}(\mathcal{A}^{\otimes 3},\mathcal{A})^1 & & \\
 & \mathrm{End}(\mathcal{A})^0 & & \mathrm{Hom}(\mathcal{A}^{\otimes 2},\mathcal{A})^1 & & \mathrm{Hom}(\mathcal{A}^{\otimes 3},\mathcal{A})^2 & \\
\mathcal{A}^0 & & \mathrm{End}(\mathcal{A})^1 & & \mathrm{Hom}(\mathcal{A}^{\otimes 2},\mathcal{A})^2 & & \mathrm{Hom}(\mathcal{A}^{\otimes 3},\mathcal{A})^3 \\
 & \mathcal{A}^1 & & \mathrm{End}(\mathcal{A})^2 & & \mathrm{Hom}(\mathcal{A}^{\otimes 2},\mathcal{A})^3 & \\
 & & \mathcal{A}^2 & & \mathrm{End}(\mathcal{A})^3 & & \\
 & & & \mathcal{A}^3 & & &
\end{array}
$$

Table 2: The relevant portion of the Hochschild complex, organized according to the Hodge diamond.

spondences suggest that the leading component for an element $\phi_I \in H^{-p,q}(M)$ is a map $\Phi_I^{(p)} \in \mathrm{Hom}(\mathcal{A}^{\otimes p},\mathcal{A})^q$ of order p and degree q. We will argue below that this is in fact true. This leads us to organize the Hochschild complex in accordance to the Hodge diamond, as depicted in table 2.

Let us motivate this more general relation. Remember that closed string operators in the BRST cohomology where functions of $z^\mu, \bar{z}^{\bar{\mu}}, \bar{\eta}^{\bar{\mu}}, \chi_\mu$. The degree p corresponds to the number of $\bar{\eta}$'s, and the degree q to the number of χ's. To find the leading component of the map in the Hochschild complex, one replaces all the χ_μ by the differential operator ∂_μ. If we start from a $(-p,q)$-form ϕ_I, defined as in (3.6), this gives a p-linear map $\Phi_I^{(p)}$ of degree q acting on the open string algebra $\mathcal{A}$. This map is given by the degree q polydifferential operator

$$
\Phi_I^{(p)} = \phi^{\mu_1\ldots\mu_p}_{\bar{\mu}_1\ldots\bar{\mu}_q}\bar{\eta}^{\bar{\mu}_1}\cdots\bar{\eta}^{\bar{\mu}_q}\,\partial_{\mu_1}\wedge\cdots\wedge\partial_{\mu_p}\,. \tag{5.11}
$$

This is indeed an element of $\mathrm{Hom}(\mathcal{A}^{\otimes p},\mathcal{A})^q$, confirming the above placement in the Hodge diamond. It can also be seen that this will correspond to the correlation function, as can be seen by fully contracting all χ_μ to z^μ in different boundary operators, therefore acting indeed as ∂_μ. These maps, for ϕ_I an element of the cohomology, can indeed be seen to be closed with respect to the coboundary δ_m. Again, there are descendants $\Phi_I^{(n)}$ for $0 \le n < p$, which can be found by replacing some χ_μ by $F^{1,1}$

$$
\Phi_I^{(n)} = \kappa^p\phi^{\mu_1\ldots\mu_p}_{\bar{\mu}_1\ldots\bar{\mu}_q}\bar{\eta}^{\bar{\mu}_1}\cdots\bar{\eta}^{\bar{\mu}_q}\bar{\eta}^{\bar{\nu}_1}\cdots\bar{\eta}^{\bar{\nu}_{p-n}}F_{\mu_1\bar{\nu}_1}\cdots F_{\mu_{p-n}\bar{\nu}_{p-n}}\partial_{\mu_{p-n+1}}\wedge\cdots\wedge\partial_{\mu_p} + \text{perms}\,. \tag{5.12}
$$

These have degrees such that the total degree in the Hochschild complex is constant, equal to $p+q$. They also satisfy the descent equations $\delta_m\Phi_I^{(n)} = -\delta_Q\Phi_I^{(n+1)}$. In particular, the leading component of order p is closed with respect to δ_m. In general, we can summarize what we said above by replacing the fermion χ_μ with a "covariant" holomorphic derivative

$$
\chi_\mu \to \kappa\frac{D}{Dz^\mu} = \kappa\frac{\partial}{\partial z^\mu} + \kappa F_{\mu\bar{\mu}}\bar{\eta}^{\bar{\mu}} + \kappa\partial_\mu X^i\chi_i\,, \tag{5.13}
$$

which is closed with respect to the total coboundary $\delta_m + \delta_Q$. We also included a term involving the scalar for completeness.

5.4 Observables and an open-closed SFT action

From the above correlation functions we can write down the gauge invariant observable (first order) $\mathcal{O}_\Phi$ for $\phi \in \bigoplus_p H^{-p,q}(M)$ for $p+q=2$ in the following form

$$\mathcal{O}_\Phi = \int \phi \cdot \Omega \wedge \mathrm{Tr}\Big(A + \frac{\kappa}{2} A \wedge \partial A + \frac{\kappa^2}{6} A \wedge \partial A \wedge \partial A\Big) = \int \phi \cdot \Omega \wedge \mathrm{Tr}(f(\kappa F^{1,1}) \wedge A)\,, \quad (5.14)$$

where in the last expression we introduced the function $f(x) = \frac{\mathrm{e}^x - 1}{x} = 1 + \frac{1}{2}x + \frac{1}{6}x^2 + \cdots$, and $f(\kappa F^{1,1})$ should be understood in terms of a Taylor expansion, using the wedge product. This function is such that $\partial(f(\kappa F^{1,1}) \wedge \kappa A) = \mathrm{e}^{\kappa F^{1,1}} - 1$.

As explained before, this observable can be understood as a first order deformation of the open string field theory action S_0. In the context of an open-closed string field theory for the B-model, we can interpret ϕ as the (physical) on-shell closed string field. The deformed action $S = S_0 + \mathcal{O}_{\bar{\Phi}}$ can then be interpreted as a first approximation to the action of the open-closed string field theory, with the closed string taken on-shell.

By inserting the different components, we recover the amplitudes calculated above. Indeed we can write the above as (at least when we expand $\star_\theta$ to first order in θ)

$$S = \int \Omega \wedge \mathrm{Tr}\left(\beta \wedge A + \frac{1}{2} A \star_\theta (\bar{\partial} + \kappa\varphi \cdot \partial) A + \frac{1}{3} A \star_\theta A \star_\theta A\right) + \mathcal{O}(\phi^2)\,, \quad (5.15)$$

with $\beta \in H^{0,2}(M)$, $\varphi \in H^{-1,1}(M)$, and $\theta = \kappa^{-1}\delta\star \in H^{-2,0}(M)$, $\phi = \beta + \varphi + \theta$. Or in other words, we have the deformed action with deformed data $Q_\varphi = \bar{\partial} + \kappa\varphi \cdot \partial$, $\star_\theta = \wedge + \kappa^2\theta \cdot \partial \wedge \partial + \mathcal{O}(\kappa^4\theta^2)$, and the same integral. Only, there is a tadpole proportional to β.

We should remark that the full action is still cubic in A, although it does not look like it from the above form. The form degrees indeed forbid more than three appearances of A in the action. This is very nontrivial, and only works because we are working in 3 complex dimensions.

For general on-shell bulk operators $\phi \in H^{-p,q}(M)$ with $p + q \leq 2$ we expect for the total action

$$\mathcal{O}_\Phi = \int \phi \cdot \Omega \wedge \mathrm{Tr}\Big(f(\kappa F^{1,1}) \wedge \Big(A + \bar{\partial} A + \frac{1}{2} A \wedge A + \frac{1}{2} A \wedge \bar{\partial} A + \frac{1}{3} A \wedge A \wedge A\Big)\Big)\,. \quad (5.16)$$

For $p + q > 2$ we should probably also consider non-physical open string operators. This is to be expected, as these operators have too high a ghost number

Let us now also include the scalars into our discussion. The two interactions discussed earlier give rise to the following two terms in the action of the open-closed string field theory

$$\sum_n \frac{\kappa^{n+1}}{(n+1)!} \int_C \Omega_{\mu ij}\, \mathrm{STr}((X^k\partial_k)^n \varphi^i_{\bar{\mu}} X^j) + \sum_n \frac{\kappa^{n+2}}{(n+2)!} \int_C \Omega_{\nu ij}\, \mathrm{STr}((X^k\partial_k)^n \varphi^\nu_{\bar{\mu}} X^i \partial_\mu X^j)\,, \quad (5.17)$$

where STr denotes the symmetrized trace. Note that the contribution of $\mathrm{e}^{\kappa X^k \partial_k}\varphi$ can be understood as a Taylor expansion, replacing the dependence of the normal coordinate z^i in φ by κX^i. So indeed, we can understand the scalar fields as describing movement in the normal direction.

A similar analysis can be done for $\theta \in H^{-2,0}(M)$. The correlators give the following two terms in the action

$$\frac{1}{2}\int_C \theta^{ij}\Omega_{zij}\,\mathrm{Tr}(A_{\bar z}) + \kappa\int_C \theta^{zi}\Omega_{zij}\,\mathrm{Tr}(A_{\bar z}\partial_z X^j)\,. \tag{5.18}$$

Ignoring the transverse derivatives of ϕ for the moment, we can summarize the first order open-closed string action in the form

$$\mathcal{O}_\Phi = \int \phi\cdot\Omega\cdot\mathrm{Tr}(f(\kappa F^{1,1}+\kappa\partial X)\wedge(A+X))\,, \tag{5.19}$$

where $\cdot$ denotes the contraction of any holomorphic vector index with a holomorphic form index, and $\wedge$ denotes wedge products both for form and vector indices.

To include the transverse derivatives, we note that there are two types of terms involving scalars X and derivatives: terms involving $X\cdot\partial\phi$ and of the form $\phi\cdot\partial X$. In fact, covariance forces them to appear in a particular combination, which we can interpret a the holomorphic Lie-derivative $\mathcal{L}_X$. This is most conveniently written in terms of action on forms. For this action we can write

$$\mathcal{L}_X = i_X\partial + \partial i_X\,, \tag{5.20}$$

giving indeed the two types of terms we found. A covariant form involving the transverse derivatives can be obtained by replacing ∂X in (5.19) with $\mathcal{L}_X$, where the derivatives are supposed to act on $\phi\cdot\Omega$.

We can give a convenient covariant description, which also shows its form as a generalized Chern-Simons term. For this, we consider a family of Calabi-Yau manifold $M\times\mathbb{C}$. The brane C is replaced by a family C_y parametrized by $y\in\mathbb{C}$, which reduces to $C_0=C$ at the origin. On this brane the open string fields are allowed to have an auxiliary dependence on the transverse coordinates z^i. At $y=0$ the boundary condition reduces to $z^i=0$.

Above we have noticed the role of the gauge field and the scalars. The effect of a background gauge field on the correlators can be summarized by the insertion of $\mathrm{e}^{\kappa F^{1,1}}$. The effect of a background scalar is to translate the brane, by the action of the Lie-derivative $\mathcal{L}_X$. We now combine the two effects into a single exponential. Using the fact that the closed string field ϕ is on shell, this exponential is actually closed, that is we have a descent-like equation

$$\mathrm{Tr}\left(\mathrm{e}^{\kappa F^{1,1}+\kappa\mathcal{L}_X}(\phi\cdot\Omega)\right) = \kappa\partial Y(\phi;A,X) + \mathrm{Tr}(\phi\cdot\Omega)\,, \tag{5.21}$$

for some Chern-Simons form Y. We could of course express Y in terms of the function f as above. To show this we use (5.20), and observe that the first term will never contribute, as all terms are ∂-closed. We can then write the observable as

$$\mathcal{O}_\Phi = \int_C Y = \int_D \mathrm{Tr}\left(\mathrm{e}^{\kappa F^{1,1}+\kappa\mathcal{L}_X}(\phi\cdot\Omega)\right) - \int_D \mathrm{Tr}(\phi\cdot\Omega)\,. \tag{5.22}$$

In the last expression we used an auxiliary chain $D\subset M\times\mathbb{C}$ such that $\partial D = C$. Note that when $C=M$ this gives the correct expression, as we remarked before. Also when there are no gauge fields in the expression (this happens when $q = \dim_{\mathbb{C}} C$), the formula can be interpreted as a translation of the brane. Writing $\omega = \phi \cdot \Omega$ the integrand can be written

$$\sum_n \frac{\kappa^n}{(n+1)!} i_X (\partial i_X)^n \omega = i_X \omega + \frac{\kappa}{2!} i_X \partial(i_X \omega) + \frac{\kappa^2}{3!} i_X \partial(i_X \partial(i_X \omega)) + \cdots . \tag{5.23}$$

which appeared in a special situation in [53, 54].

6. The Hochschild cohomology of HCS

In this section we explicitly calculate the Hochschild cohomology of the open string algebra of the B-model. To do the calculation, we will use Hochschild-Kostant-Rosenberg theorem reviewed in appendix A.3.

6.1 Calculation of the Hochschild cohomology

We take for the open string algebra the full off-shell algebra $\mathcal{A} = \Omega^{0,*}(M, \mathrm{End}(E))$ of endomorphism valued $(0,p)$-forms. This has the structure of a differential graded associative algebra (dg-algebra), with differential $\bar{\partial}_A$ and the product is the combination of the wedge product and the local matrix product in $\mathrm{End}(E)$.

To see what happens we will first take a look at the case where E is the trivial U(1) bundle, i.e. we take $\mathcal{A} = \Omega^{0,*}(M)$. Later we shall argue that we can reduce to this case also in the situation of a nontrivial bundle, making use of the Morita equivalence between these algebras.

We first look at the most trivial case of a flat CY manifold $M = \mathbb{C}^3$ with a trivial bundle, so that $\bar{\partial}_A = \bar{\partial}$. We can reduce the problem to the much better behaved problem of Hochschild cohomology of a polynomial algebra. Note that the polynomial forms are dense in the algebra. So we replace the algebra $\mathcal{A}$ by the polynomial algebra $\mathcal{A} = \mathbb{C}[z^\mu, \bar{z}^{\bar{\mu}}, \bar{\eta}^{\bar{\mu}}]$, where z^μ and $\bar{z}^{\bar{\mu}}$ are generators of degree 0, and $\bar{\eta}^{\bar{\mu}}$ is a Grassmann generator if degree 1. The differential on this algebra can be written $Q = \bar{\eta}^{\bar{\mu}} \frac{\partial}{\partial \bar{z}^{\bar{\mu}}}$. If this differential were zero, the Hochschild cohomology would be the cohomology of the multilinear maps with respect to the Hochschild coboundary δ_m, $HH^*(\mathcal{A}) = H^*_{\delta_m}(C^*(\mathcal{A}, \mathcal{A}))$. This is precisely the situation handled by the Hochschild-Kostant-Rosenberg theorem, giving the polynomial algebra

$$\mathcal{C} = H^*_{\delta_m}(C^*(\mathcal{A}, \mathcal{A})) = \mathbb{C}[z^\mu, \bar{z}^{\bar{\mu}}, \bar{\eta}^{\bar{\mu}}; \chi_\mu, \bar{\chi}_{\bar{\mu}}, \bar{p}_{\bar{\mu}}] , \tag{6.1}$$

where the extra generators χ_μ and $\bar{\chi}_{\bar{\mu}}$ have degree 1, and the generators $\bar{p}_{\bar{\mu}}$ have degree 0. The extra generators can be understood as conjugate to the original generators of $\mathcal{A}$. In the Hochschild complex of multilinear maps, they correspond to the differential operators $\partial/\partial z^\mu$, $\partial/\partial \bar{z}^{\bar{\mu}}$ and $\partial/\partial \bar{\eta}^{\bar{\mu}}$, respectively. The conjugate relation can also be stated in terms of the Gerstenhaber structure of the Hochschild cohomology. This is endowed with an odd Poisson bracket, the Gerstenhaber bracket, which is given by the bidifferential operator

$$\frac{\partial}{\partial z^\mu} \wedge \frac{\partial}{\partial \chi_\mu} + \frac{\partial}{\partial \bar{z}^{\bar{\mu}}} \wedge \frac{\partial}{\partial \bar{\chi}_{\bar{\mu}}} + \frac{\partial}{\partial \bar{\eta}^{\bar{\mu}}} \wedge \frac{\partial}{\partial \bar{p}_{\bar{\mu}}} . \tag{6.2}$$

When we take into account the differential Q on the algebra $\mathcal{A}$, we get an induced differential on the algebra $\mathcal{C}$, which we called δ_Q before. It is given on the above polynomial algebra by

$$\delta_Q = \bar{\eta}^{\bar{\mu}} \frac{\partial}{\partial \bar{z}^{\bar{\mu}}} + \bar{\chi}_{\bar{\mu}} \frac{\partial}{\partial \bar{p}_{\bar{\mu}}} \equiv \delta_1 + \delta_2 \,. \tag{6.3}$$

To calculate the total Hochschild cohomology $H^*(\mathrm{Hoch}(\mathcal{A}))$ we need to take the cohomology with respect to this differential.[2] Corresponding to the two factors, we write the above algebra as a tensor product

$$\mathcal{C} = \mathbb{C}[z^\mu, \bar{z}^{\bar{\mu}}, \bar{\eta}^{\bar{\mu}}, \chi_\mu] \otimes \mathbb{C}[\bar{\chi}_{\bar{\mu}}, \bar{p}_{\bar{\mu}}] \,. \tag{6.4}$$

The idea is that the second factor is always trivial. The reason is that the variables $\bar{\chi}_{\bar{\mu}}, \bar{p}_{\bar{\mu}}$ always generate a vector space, as they are coordinates on the fiber of the twisted holomorphic cotangent space to the $\bar{z}^{\bar{\mu}}, \bar{\eta}_{\bar{\mu}}$-space. This should be contrasted with the first factor, where the generators $z^\mu, \bar{z}^{\bar{\mu}}$ should be interpreted as local coordinates on a topologically nontrivial manifold. The above polynomial algebra is only a local description of the full algebra $\mathcal{C}$. As the two terms in the coboundary act on each factor separately, the cohomology is given by the tensor product of the cohomologies of the two factors. The second factor is easily seen to be completely trivial, and is given by[3]

$$H^*_{\delta_2}(\mathbb{C}[\bar{\chi}_{\bar{\mu}}, \bar{p}_{\bar{\mu}}]) = \mathbb{C}. \tag{6.5}$$

So we loose the second factor in (6.4), and keep only the first factor. Also, we still have to take cohomology with respect to δ_1. The total Hochschild cohomology therefore becomes

$$H^*_{\delta_Q}(\mathcal{C}) = H^*_{\delta_1}(\mathbb{C}[z^\mu, \bar{z}^{\bar{\mu}}, \bar{\eta}^{\bar{\mu}}, \chi_\mu]) \,. \tag{6.6}$$

An argument similar to the one above would have us project out the $\bar{z}^{\bar{\mu}}$ and $\bar{\eta}^{\bar{\mu}}$ dependence. This would therefore not reproduce the full on-shell closed string algebra. But we should realize that the polynomial algebra only gives a local picture, in a contactable coordinate chart. And indeed the the cohomology in a local chart is expected to be trivial. Globally, one should find nontrivial cohomology.

We can give a more precise global description as follows. The bosonic part is generated by the coordinates $z^\mu, \bar{z}^{\bar{\mu}}, \bar{p}_{\bar{\mu}}$, which are coordinates on the total space of $\overline{\mathcal{T}}^*_M$. Let us denote the projection on M by $\pi : \overline{\mathcal{T}}^*_M \to M$. The algebra $\mathcal{C}$ can be described as sections of complex polyvector fields on the total space of $\overline{\mathcal{T}}^*_M$,

$$\mathcal{C} = \Gamma(\textstyle\bigwedge T(\overline{\mathcal{T}}^*_M) \otimes \mathbb{C}) = \Gamma(\bigwedge \pi^* \mathcal{T}_M \otimes \bigwedge \pi^* \overline{\mathcal{T}}_M \otimes \bigwedge \pi^* \overline{\mathcal{T}}^*_M) \,. \tag{6.7}$$

Here we used that the vertical tangent space to $\overline{\mathcal{T}}^*_M$ can be identified with $\pi^* \overline{\mathcal{T}}^*_M$. This space can now be written as a tensor product of algebras over the algebra $\mathcal{O}_M$ of complex functions on M,

$$\Gamma(\textstyle\bigwedge \mathcal{T}_M \otimes \bigwedge \overline{\mathcal{T}}_M) \otimes_{\mathcal{O}_M} \Gamma(\bigwedge \pi^* \overline{\mathcal{T}}^*_M) \,. \tag{6.8}$$

[2]A more precise procedure would involve a spectral sequence calculation for the double complex $\mathrm{Hoch}(\mathcal{A})$ with the total differential $\delta_m + \delta_Q$, for which the sketched procedure gives the second term.

[3]This can be realized by viewing it as the complexification of the cohomology of $\mathbb{R}^3$, parametrized by $\bar{p}_{\bar{\mu}}$, and $H^*(\mathbb{R}^3) = \mathbb{R}$ (supported at degree zero).

The second part of the coboundary, δ_2, acts only on the second part of this tensor product and commutes with the action of $\mathcal{O}_M$. The correct way to calculate the total cohomology with respect to $\delta_Q = \delta_1 + \delta_2$ would be to view the above again as a double complex, with the two terms as the two differentials, which we can calculate using spectral sequence techniques. The first term of this spectral sequence is the cohomology with respect to δ_2, and as $H^*_{\delta_1}(\bigwedge \pi^* \overline{\mathcal{T}}^*_M) = \mathcal{O}_M$ this is simply the first factor in the tensor product above,

$$E_1^{*,*} = \Gamma(\bigwedge \mathcal{T}_M \otimes \bigwedge \overline{\mathcal{T}}_M) = \Omega(M, \bigwedge \mathcal{T}_M)\,. \tag{6.9}$$

The remaining coboundary is δ_1, which is identified with the (twisted) Dolbeault operator $\bar\partial$ acting on this space. Therefore, the second term in the spectral sequence now gives the required cohomology

$$H^*(\mathrm{Hoch}(\mathcal{A})) = H^*_{\bar\partial}(M, \bigwedge^* \mathcal{T}_M)\,, \tag{6.10}$$

precisely the space of on-shell closed string operators. The sequence terminates, as we can always choose representatives that are independent of $\bar\chi$ and $\bar p$.

We should note that the cohomology with respect to δ_m can be performed reliably in a local chart, due to the fact that the product is local. It is also important to note that indeed the cohomology of the second factor decouples globally. This is due to the fact that it involves the fibers only, which are globally trivial. A more precise argument would give the cohomology as the first term of a spectral sequence for the double complex formed by the terms δ_2 and δ_1. Because the δ_2 cohomology is trivial, there is no room for descent equations, so we need not go further than the second term.

If we consider the situation with a nontrivial gauge bundle E and holomorphic connection $\bar\partial_A$. In fact, the algebra with values in $\mathrm{End}(E)$ is Morita equivalent to the algebra with trivial bundle. Furthermore, it is a well known result that the Hochschild cohomology is invariant under Morita equivalence. Therefore, the result does not depend on the gauge bundle.

6.2 Identification of cohomologies and formality

We see that the Hochschild cohomology $H^*(\mathrm{Hoch}(\mathcal{A}))$ is precisely given by the BRST cohomology of the closed string theory, namely they can both be identified with the algebra $H^*_{\bar\partial}(M, \bigwedge^* \mathcal{T}_M)$. This gives confidence to our conjecture that in general the Hochschild cohomology calculates the on-shell closed string.

This fact is actually important for the formality conjecture, in relation to path integral representations of this formality map, as in [19, 1]. There it was implicitly assumed for formality that the two cohomologies are the same. Notice that formality is a statement about the relation between the Hochschild complex and its cohomology (namely, that they are quasi-isomorphic as L_∞ algebras). On the other hand, the path integral gives a representation of a map from the closed string BRST cohomology to the Hochschild cohomology (an action on the A_∞ algebra of the boundary theory). This can only be interpreted as formality if the cohomologies are the same.

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Formality of a complex (as a particular type of algebra) means that it is quasi-isomorphic to its cohomology.This means that there should be an intertwining map between the cohomology and the complex (or the other way around) that becomes an isomorphism in cohomology.

More concretely, we have constructed a map from the closed string BRST cohomology to the Hochschild complex Hoch($\mathcal{A}$), as $\phi \mapsto \Phi$. This map is intertwining as shown in [1]. With the above calculation of the Hochschild cohomology we have shown that this map reduces to an isomorphism on the cohomology, as Gerstenhaber algebras algebras. In mathematical terms this means that the closed string BRST cohomology is quasi-isomorphic to the Hochschild complex. As the former can be identified with the cohomology of the latter, this reduces precisely to the mathematical notion of formality.

This map $\phi \mapsto \Phi$ is only the first order approximation, in the context of the Hochschild cohomology. This can be seen by the fact that it is intertwining only to lowest order. Under certain conditions it can be extended to the full formality map $\phi \mapsto \bar{\Phi}$ which contains all higher order correction in the closed string field ϕ. The components $\bar{\Phi}^{(n)}$ can be calculated similarly from the sigma model as the $n+1$-point functions completely deformed by ϕ. These maps satisfy the full nonlinear master equation (2.12).

In general, there is an obstruction to extend the first order solution Φ in the Hochschild cohomology to a full solution $\bar{\Phi}$ of the master equation. Formality of the Hochschild complex says that there is a quasi isomorphism between the Hochschild cohomology, which has been shown to be the closed string BRST cohomology, and the Hochschild complex. This is a quasi isomorphism of L_∞ algebras (and more generally, of G_∞ algebras [16]). It therefore maps solutions of the master equation in the cohomology to solutions in the complex. As $\delta = \delta_Q + \delta_m$ is zero in the cohomology, the master equation there reduces simply to $\{\phi, \phi\} = 0$. Hence this is a necessary condition for a full solution $\bar{\Phi}$ to exist.

In the B-model, the master equation can be written in terms of the forms as

$$\frac{\partial \phi}{\partial z^\mu} \frac{\partial \phi}{\partial \chi_\mu} = 0\,. \tag{6.11}$$

This is nothing but the part that remains from the Gerstenhaber bracket (6.2). For a deformation of the complex structure $\varphi \in H^{-1,1}(M)$ this becomes

$$\partial_\nu \varphi^\mu_{[\bar{\mu}} \varphi^\nu_{\bar{\nu}]} = 0\,. \tag{6.12}$$

This can be understood as the quadratic part of $(\bar{\partial} + \varphi \cdot \partial)^2 = 0$. For an element $\theta \in H^{-2,0}(M)$ the master equation can be written

$$\theta^{\rho[\lambda} \partial_\rho \theta^{\mu\nu]} = 0\,. \tag{6.13}$$

This equation says that θ is a holomorphic Poisson structure. It is the condition for $\star_\theta$ to be associative to lowest order.

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7. A BV sigma-model

In this section we present a BV sigma-model giving an off-shell formulation for the B-model. This model is inspired by the calculation of the Hochschild cohomology in the previous section.

7.1 The BV model

Above, we found the Hochschild cohomology of the open string B-model. In the intermediate step we found the algebra $\mathcal{C}$ which was still provided with a nontrivial differential δ_Q. Also, we saw that this algebra had a natural Gerstenhaber structure. Actually, this Gerstenhaber structure is easily seen to be part of a BV structure. That is, the bracket can be derived from the BV operator

$$\triangle = \frac{\partial^2}{\partial z^\mu \partial \chi_\mu} + \frac{\partial^2}{\partial \bar{z}^{\bar{\mu}} \partial \bar{\chi}_{\bar{\mu}}} + \frac{\partial^2}{\partial \bar{\eta}^{\bar{\mu}} \partial \bar{p}_{\bar{\mu}}}, \tag{7.1}$$

as the failure of derivation condition.

This BV structure can be used to define a 2-dimensional BV sigma model. To this end we introduce supercoordinates $(x^\alpha|\theta^\alpha)$ on the super worldsheet $\Pi T\Sigma$, where the x^α are the bosonic degree 0 coordinates on Σ and the θ^α are fermionic degree 1 Grassmann coordinates on the fiber. The superfields of the sigma model are superfields — functions of the supercoordinates above — corresponding to the generators of $\mathcal{C}$. We will denote superfields using bold characters; they are given by

$$\begin{aligned} \boldsymbol{z}^\mu &= z^\mu + \theta\rho^\mu + \theta^2 f^\mu, & \boldsymbol{\chi}_\mu &= \chi_\mu + \theta q_\mu + \theta^2\zeta_\mu, \\ \bar{\boldsymbol{z}}^{\bar{\mu}} &= \bar{z}^{\bar{\mu}} + \theta\bar{\rho}^{\bar{\mu}} + \theta^2 \bar{f}^{\bar{\mu}}, & \bar{\boldsymbol{\chi}}_{\bar{\mu}} &= \bar{\chi}_{\bar{\mu}} + \theta\bar{q}_{\bar{\mu}} + \theta^2\bar{\zeta}_{\bar{\mu}}, \\ \bar{\boldsymbol{p}}_{\bar{\mu}} &= \bar{p}_{\bar{\mu}} + \theta\bar{\xi}_{\bar{\mu}} + \theta^2\bar{r}_{\bar{\mu}}, & \bar{\boldsymbol{\eta}}^{\bar{\mu}} &= \bar{\eta}^{\bar{\mu}} + \theta\bar{h}^{\bar{\mu}} + \theta^2\bar{\pi}^{\bar{\mu}}. \end{aligned} \tag{7.2}$$

Here we did not explicitly write worldsheet form and vector indices and their contractions. On the space of superfields we have a BV structure induced by the BV operator $\triangle$, and a corresponding BV antibracket $(\cdot,\cdot)_{BV}$. The BV action of the sigma-model will be given by

$$S = \frac{1}{\kappa}\int_{\Pi T\Sigma} d^2x d^2\theta (\boldsymbol{\chi}_\mu D\boldsymbol{z}^\mu + \bar{\boldsymbol{\chi}}_{\bar{\mu}} D\bar{\boldsymbol{z}}^{\bar{\mu}} + \bar{\boldsymbol{p}}_{\bar{\mu}} D\bar{\boldsymbol{\eta}}^{\bar{\mu}} + \bar{\boldsymbol{\chi}}_{\bar{\mu}}\bar{\boldsymbol{\eta}}^{\bar{\mu}}), \tag{7.3}$$

where $D = \theta^\alpha\partial_\alpha$. The kinetic term in fact expresses the canonical relations between the generators. The last term induces the nontrivial differential on the $\mathcal{C}$. In the BV language, this is the BRST operator, which is given by

$$\boldsymbol{Q} = (S,\cdot)_{BV} = D + \bar{\boldsymbol{\eta}}^{\bar{\mu}}\frac{\partial}{\partial\bar{\boldsymbol{z}}^{\bar{\mu}}} + \bar{\boldsymbol{\chi}}_{\bar{\mu}}\frac{\partial}{\partial\bar{\boldsymbol{p}}_{\bar{\mu}}}. \tag{7.4}$$

This should be compared to the form of δ_Q (6.3). In fact the potential term in the action was chosen precisely to reproduce this form. The last part of the structure of the BV sigma model is the 1-form operator $\boldsymbol{G}_\alpha = \partial/\partial\theta^\alpha$. It corresponds to the operator G_α, and satisfies the analogous relation $\{\boldsymbol{Q},\boldsymbol{G}\} = d$.

On the open worldsheet, the fields $\boldsymbol{z}^\mu, \bar{\boldsymbol{z}}^{\bar\mu}, \bar{\boldsymbol{\eta}}^{\bar\mu}$ will satisfy Neumann like boundary conditions, while the fields $\boldsymbol{\chi}_\mu, \bar{\boldsymbol{\chi}}_{\bar\mu}, \bar{\boldsymbol{p}}_{\bar\mu}$ satisfy Dirichlet boundary conditions. This implies that boundary operators are generated by functions of the first set of coordinates. These are precisely the observables of the open B-model. The observables in the bulk will be generated by functions of all superfields. These are pull backs to function space of the algebra $\mathcal{C}$. The BRST cohomology of bulk operators will be related to the cohomology of $\mathcal{C}$, which as we have seen precisely are the operators of the closed B-model.

The gauge fixing above shows that operators have all kind of curvature corrections. It would be interesting to see if these survive the correlators. This would be somehow strange, as the B-model is supposed to be independent of the metric.

This BV sigma model discussed in this section is closely related to similar BV models for the B-models given in [55, 56]. In fact they can be shown to be equivalent after partial gauge fixings.

7.2 Gauge fixing

The BV sigma model defined this way can be seen to be equivalent to the usual B-model in BRST quantization, given by (3.1). To see this, we need to gauge fix the BV sigma model. To gauge fix we first need to make a division of the BV fields into "fields" and "antifields". We choose for the fields all of the scalars and the one-forms ρ^μ, $\bar q_{\bar\mu}$, $\bar h^{\bar\mu}$. To gauge fix the antifields in terms of the field, we use the following gauge fixing fermion,

$$\Psi = \int_\Sigma \left(\kappa t g_{\mu\bar\mu} \rho^\mu * d\bar z^{\bar\mu} - \frac{1}{2}\Gamma^\lambda_{\mu\nu}\rho^\mu\rho^\nu\chi_\lambda \right). \tag{7.5}$$

This gives the gauge conditions

$$\begin{aligned} q_\mu &= \kappa t g_{\mu\bar\mu} * d\bar z^{\bar\mu} - \Gamma^\lambda_{\mu\nu}\rho^\nu\chi_\lambda\,, & f^\mu &= -\frac{1}{2}\Gamma^\mu_{\nu\lambda}\rho^\nu\rho^\lambda\,, \\ \bar\zeta_{\bar\mu} &= -\kappa t g_{\mu\bar\mu} d * \rho^\mu - \frac{1}{2}R^\lambda_{\mu\bar\mu\nu}\rho^\mu\rho^\nu\chi_\lambda\,, & \zeta_\mu &= \kappa t \partial_\mu g_{\nu\bar\mu}\rho^\nu * d\bar z^{\bar\mu} - \frac{1}{2}\partial_\mu\Gamma^\lambda_{\nu\tau}\rho^\nu\rho^\tau\chi_\lambda\,, \end{aligned} \tag{7.6}$$

with the other antifields vanishing. Inserting these defines the gauge fixed action. Taking the equations of motion for the auxiliary fields $\bar q_{\bar\mu}$ and $\bar p_{\bar\mu}$, we find $\bar h^{\bar\mu} \approx d\bar z^{\bar\mu}$. Substituting this in the gauge fixed action, we find back the original BRST action (3.1). Also, the operators $\boldsymbol{Q}$ and $\boldsymbol{G}$ reduce to the operators Q and G of the B-model respectively after gauge fixing. The covariantizing term in the gauge fixing term is similar to the one that can be included in the Cattaneo-Felder model [57].

Notice that the parameter t originates from the gauge fixing. This gives another explanation why the theory does not depend on t, while κ is a nontrivial coupling.

The BV sigma model can also be used to do the calculations in this paper. Due to the natural symplectic structure the formulas are much more intuitive The advantage of the BV model over the usual B-model is that the symmetry algebra generated by the BRST operator and G closes off-shell. This is not true for the original model; for example we have $\{Q, G\}\chi_\mu = \kappa t g_{\mu\bar\mu} * d\bar\eta^{\bar\mu}$, which equals $d\chi_\mu$ only on-shell. Therefore, the BV sigma model can also be used to calculate off-shell amplitudes. This allows us to check that the expressions for the fundamental operations $\Phi^{(n)}_I$ remain correct for off-shell open string operators.

7.3 Higher orders and Kontsevich's formula

The BV action allows us to calculate correlators with more than one closed string insertions in a consistent perturbative expansion. Actually, our model is very close to the Cattaneo-Felder model [19] used to calculate the star-product in deformation quantization. The perturbative expansion of their model essentially reproduced Kontsevich's formula for deformation quantization [14]. The formula Kontsevich gave is a sum over graphs, which are identified with the Feynman graphs of the CF model, where each graph represents a particular term in the expansion, and has a particular calculable weight. The weight is essentially the value of the corresponding Feynman integral. In our case, adopting the gauge fixing procedure of Cattaneo-Felder (rather than the one above), we get essentially the same propagators and similar vertices. Hence the values of the Feynman integrals, and therefore the weights of the graphs, as in their model. The conclusion is that the perturbative expansion of the open-closed string field theory is essentially given by Kontsevich's formula. For example, this shows that indeed the nonlinear coupling of the $(-2,0)$ part θ of the closed string field combines to the deformed star-product $\star_\theta$, given by the (complexification of) Kontsevich's formula.

8. Noncommutative geometry of the B-model

In this section we will interpret the results in terms of noncommutative geometry. It will be shown that the closed string algebra can be identified with with the cycles and chains in the sense of noncommutative geometry.

8.1 Cyclic cohomology

Open string theories always have an inner product, which is defined by the 2-point functions, which we assume to be nondegenerate. Therefore, we can relate multilinear maps with multilinear forms. We will denote the latter with $\Psi_I^{(n)}:\mathcal{A}^{\otimes(n+1)}\to\mathbb{C}$ to distinguish them from $\Phi_I^{(n)}:\mathcal{A}^{\otimes n}\to\mathcal{A}$. They are defined by

$$\Psi_I^{(n)}(\hat\alpha_{a_0},\hat\alpha_{a_1},\dots,\hat\alpha_{a_n})=(-1)^{|\phi||\alpha_{a_0}|}\left\langle\hat\alpha_{a_0}\Phi_I^{(n)}(\hat\alpha_{a_1},\dots,\hat\alpha_{a_n})\right\rangle=(-1)^{|\phi||\alpha_{a_0}|}\Phi^{(n)}_{Ia_0a_1\dots a_n}\,. \tag{8.1}$$

Note that actually the forms are more natural than the maps, as they do not use the inner product.

Open string correlators are in general cyclically antisymmetric, due to the fact that the boundary is closed. This will be true when we insert the closed string operator ϕ_I provided we take $\partial\phi_I=0$. In other words, the correlators $\Psi_I^{(n)}$ are (graded) cyclically antisymmetric,

$$\Psi_I^{(n)}(\hat\alpha_{a_0},\dots,\hat\alpha_{a_n})=(-1)^{n+\sum_{i>0}|\alpha_{a_0}||\alpha_{a_i}|}\Psi_I^{(n)}(\hat\alpha_{a_1},\dots,\hat\alpha_{a_n},\hat\alpha_{a_0})\,. \tag{8.2}$$

One would therefore expect that the cohomology of these maps is the cyclic cohomology of the algebra $\mathcal{A}$ rather than the Hochschild cohomology. The cyclic cohomology is a subcomplex of the Hochschild complex, consisting of the cyclically symmetric forms Ψ.

For the trivial algebra $\mathcal{A} = \mathbb{C}$, the cyclic cohomology can be found as follows. Let e be the unit element generating the algebra $\mathcal{A} = \mathbb{C}$. Then an element of cyclic cohomology is determined by the value $\Psi(e, \ldots, e) \in \mathbb{C}$. Cyclic symmetry restricts these to be zero for odd degrees n (that is, an even number of arguments), therefore the cyclic cohomology is given by $\mathbb{C}$ in each even degree. In fact, it is generated as a polynomial algebra by a single element σ of degree 2, $HC^*(\mathbb{C}) \cong \mathbb{C}[\sigma]$. If e is the unit element generating the algebra $\mathcal{A} = \mathbb{C}$, the cyclic form is given by $\sigma(e, e, e) = 1$.[4] This is however not the closed string algebra, so indeed this is not the correct identification. The cyclic cohomology is however very close to the Hochschild cohomology. The relation can be given in terms of an exact triangle as follows [10]

$$\begin{tikzcd} & HH^*(\mathcal{A}) \arrow[dl, "B"'] & \\ HC^*(\mathcal{A}) \arrow[rr, "S"] & & HC^*(\mathcal{A}) \arrow[ul, "I"'] \end{tikzcd} \tag{8.3}$$

Here I is the inclusion of the cyclic complex in the Hochschild complex. The map B is the "boundary map", which is defined by

$$B\Psi(\alpha_{a_0}, \ldots, \alpha_{a_n}) = \frac{1}{n+1}\Big(\Psi(1, \alpha_{a_0}, \ldots, \alpha_{a_n}) + (-1)^n \Psi(\alpha_{a_0}, \ldots, \alpha_{a_n}, 1)\Big) \pm \text{cycl.} \tag{8.4}$$

Notice that any (unital) algebra contains the algebra $\mathbb{C}$ as a subalgebra. Therefore, the element σ naturally is a canonical element of the cyclic cohomology of any algebra $\mathcal{A}$. The map S is the cup product with this element $S\Psi = \Psi \cup \sigma$. Note that B lowers the degree by one and S raises the degree by two. Therefore, in going around the triangle once, the degree is raised by one. The action of S on the maps Φ can be written as follows

$$\begin{aligned} &\Big\langle \hat{\alpha}_{a_0}\, (S\Phi_I)^{(n)}(\hat{\alpha}_{a_1}, \cdots, \hat{\alpha}_{a_n}) \Big\rangle = \\ &\qquad = \sum_i \frac{\pm 1}{n+1} \left\langle \hat{\phi}_I \hat{\alpha}_{a_0} \int \hat{\alpha}^{(1)}_{a_1} \cdots \int \hat{\alpha}^{(1)}_{a_{i-1}} \hat{\alpha}_{a_i} \hat{\alpha}_{a_{i+1}} \int \hat{\alpha}^{(1)}_{a_{i+2}} \cdots \int \hat{\alpha}^{(1)}_{a_n} \right\rangle \pm \text{cycl.} \end{aligned} \tag{8.5}$$

Note that the therefore we included more general correlators. However, these correlators with less descendants will be factorizable. Therefore, they do not contain new information about the operators ϕ_I. Furthermore it can be shown, compare [10], that $S\Phi$ is trivial in the Hochschild cohomology. Therefore, it corresponds to a BRST exact closed string operator.

The boundary map B can be related with the BV operator in the closed string string theory. In fact, for the B-model this operator is identified with the ∂ operator; that is it acts on the closed string operators ϕ we use as $(\triangle\phi) \cdot \Omega = \partial(\phi \cdot \Omega)$. This can explicitly be seen from the expression we had in the calculation of the Hochschild cohomology. As the closed string operators can be taken independent of $\bar{\chi}$ and $\bar{p}$, the only term that survives is $\partial^2/\partial\chi_\mu \partial z^\mu$, which is easily seen to exactly provide the above. Note that on the on-shell representatives we always can take $\triangle\phi = 0$, and we actually assumed this. If we do not

[4]This is exact in the Hochschild cohomology, as we can write $\sigma = \delta_m \tau$ where $\tau(e, e) = -1$. This immediately implies that it is closed also in the cyclic complex.

assume this, we indeed find that the explicit expression

$$\int \phi \cdot \Omega \wedge \mathrm{Tr}(\alpha_{a_0} \wedge \partial\alpha_{a_1} \wedge \cdots \wedge \partial\alpha_{a_n}) \tag{8.6}$$

for the observables are not cyclically symmetric. Indeed we have to do a partial integration to move the ∂ from α_{a_n} to α_{a_0}. In general, we get an extra term involving precisely $\partial(\phi\cdot\Omega)$. With the identification $B = (-1)^{p+q+1}\triangle$, we can write this as

$$\left\langle \alpha_{a_n}\alpha_{a_0}, B\Phi(\alpha_{a_1},\cdots,\alpha_{a_{n-1}})\right\rangle = \pm\left\langle \alpha_{a_0}, \Phi(\alpha_{a_1},\cdots,\alpha_{a_n})\right\rangle \pm \left\langle \alpha_{a_n}, \Phi(\alpha_{a_0},\cdots,\alpha_{a_{n-1}})\right\rangle . \tag{8.7}$$

Taking $\alpha_{a_n} = 1$ this exactly reproduces the definition of the boundary operator B above. It can also be checked that with the identification of B as above also the signs are exactly reproduced.

In the correlation functions, the reason that the correlation functions are not cyclic when $\triangle\phi \neq 0$ comes from a correction in the Ward identity. Remember that the descent operator G is generated by a current $\hat{b}$. The general action of this current on a closed string operator $\hat{\phi}$ is given by

$$\oint_{C_z} \xi(w)\hat{b}(w)\,\hat{\phi}(z) = \xi(z)\hat{\phi}^{(1)}(z) + \xi'(z)(\triangle\hat{\phi})(z)\,, \tag{8.8}$$

where $\xi(z)$ is a vector on the worldsheet. In proving the Ward identity giving the cyclicity, we used a vector field ξ vanishing at the point z where the closed string operator was inserted.

The relation between the cyclic cohomology and the Hochschild cohomology suggests the following interpretation in terms of the open string theory. The elements of the Hochschild cohomology correspond to the irreducible string diagrams, while the cyclic cohomology in general contains diagrams that can be factorized. The action on $\hat{\phi}$ will therefore vanish provided we take $\triangle\hat{\phi} = 0$. If this is not the case, there will be a correction as $\xi'(z)$ can not be chosen to vanish. This correction will generate the extra term involving B.

Next we remark on the structure of the observables. Note that we can write it in the form

$$\sum_{n\geq 1} \frac{1}{(n+1)!}\int \Omega\cdot \mathrm{Tr}(A(\triangle A)^n\phi) + \cdots\,, \tag{8.9}$$

where the dots denote the higher order corrections in ϕ, and we should interpret A as the full open string field including both the gauge field and the scalar. This expression can be interpreted as a potential for the deformation.

8.2 Dirac operator and spectral triples

In our discussion on gauge invariant observables, we could think of the closed string operators ϕ as cycles over which we integrate the abstract 1-form A. Actually, we can make this correspondence more exact using the language of noncommutative geometry [10]. In noncommutative geometry, cycles are elements of the cyclic cohomology. This is closely related to the Hochschild cohomology. We already saw above that the mixed correlators could be understood as elements of the cyclic cohomology. And therefore, the Φ_I can indeed be interpreted as cycles in the cyclic cohomology of the open string algebra $\mathcal{A}$. In noncommutative geometry the cycles can be represented by characters, given by an integral $\int$ on a graded algebra Ω supplied with a map $\rho : \mathcal{A} \to \Omega^0$ and a differential d, as

$$\tau(a_0, a_1, \dots, a_n) = \int \rho(a_0) d\rho(a_1) \cdots d\rho(a_n) \,. \tag{8.10}$$

In the language of correlation functions, we can identify d with G, Ω with the algebra generated by $\alpha_0 \alpha_1^{(1)} \cdots \alpha_n^{(1)}$, and the integral with the correlator with a ϕ insertion and an iterated integral over the boundary. The map ρ is the obvious inclusion.

More generally, Connes supplied a notion of metric on the space using a Dirac operator. What is needed is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, consisting of an algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ and a Dirac operator D. The algebra $\mathcal{A}$ and the Dirac operator D act on the Hilbert space $\mathcal{H}$, and there is the condition that $[D, a]$ is bounded for any $a \in \mathcal{A}$ and D has its spectrum in $\mathbb{R}$.

We will now show that the B-model open string has a natural spectral triple. We already know that $\mathcal{A} = \Omega^{0,*}(M, \mathrm{End}(E))$. Next we have to supply the Hilbert space $\mathcal{H}$ and the Dirac operator D. As $\mathcal{A}$ has to act on the Hilbert space, it is natural to take for $\mathcal{H}$ the Hilbert space of L^2 sections of E-valued differential forms. At first, it seems that we could take $(0, q)$-forms. This is indeed enough for the differential geometry, but as we will see in order to have a Dirac operator we should take any (p, q)-form. A natural construction of the Dirac operator is to use a spinor bundle, and D the usual Dirac operator on this spinor bundle. Let us therefore try to use the spinor bundle on M. As M is a Kähler manifold (indeed, we assume it to be Calabi-Yau) a natural spinor bundle is provided by the bundle $\mathcal{S} = \bigwedge^* \mathcal{T}_M^*$ of $(p, 0)$-forms. The Hilbert space and the Dirac operator are then given by

$$\mathcal{H} = \Omega^{0,*}(M, \mathcal{S} \otimes E) = \Omega^{*,*}(M, E) \,, \qquad D = \partial + \partial^\dagger \,, \tag{8.11}$$

where ∂ and $\partial^\dagger$ are the Dolbeault operators twisted by the bundle E, and the Hilbert space should be understood as a space of L^2 sections. The spinor bundle, and therefore also the Hilbert space $\mathcal{H}$, has a natural $\mathbb{Z}_2$ grading induced by the degree. The Dirac operator is odd with respect to this degree. Note that indeed with an appropriate regularity for $\mathcal{A}$ we have that $[D, \mathcal{A}]$ is a subalgebra of the bounded operators on $\mathcal{H}$.

In noncommutative geometry, the Dirac operator can be used to recover the differential geometry. This is a particular representation of differential forms as

$$\alpha_0 [D, \alpha_1] \cdots [D, \alpha_n] \,. \tag{8.12}$$

In fact one has to consider classes of them. Then cycles can be defined in terms of traces of these elements,

$$\mathrm{Tr}_\omega (\alpha_0 [D, \alpha_1] \cdots [D, \alpha_n]) \,. \tag{8.13}$$

Note that in the B-model we had a correspondence between operators in the string theory and the geometrical operations as

$$Q \sim \bar{\partial} \,, \qquad Q' \sim \partial^\dagger \,, \qquad G_{||} \sim \partial \,, \qquad G_\perp \sim \bar{\partial}^\dagger \,. \tag{8.14}$$

This came by identifying $\bar{\eta}^{\bar{\mu}}$ with the basic $(0,1)$-forms $d\bar{z}^{\bar{\mu}}$, and ρ^μ with the basic $(1,0)$-forms dz^μ. We then see that the Dirac operator corresponds to the worldsheet operator $D \sim G_{||} + Q'$.

9. Other models

We can do the same calculation for other topological string theories. We discuss here the trivial open string (with only the Chan-Paton degrees of freedom) the A-model and the Cattaneo-Felder model (C-model).

9.1 The trivial model

We start with the trivial model, whose only degrees of freedom are the Chan-Paton indices. The open string algebra is simply the matrix algebra $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C}) = \mathrm{End}(\mathbb{C}^N)$. This algebra is Morita equivalent to the algebra $\mathbb{C}$. As the Hochschild cohomology is invariant under Morita equivalence, this implies that $H^*(\mathrm{Hoch}(\mathcal{A})) = H^*(\mathrm{Hoch}(\mathbb{C})) = \mathbb{C}$. This is indeed the on-shell closed string algebra, as the only operators of the closed string are multiples of the identity.

9.2 The A-model

We next consider the topological A-model. The A-model is defined for a lagrangian 3-cycle in a Calabi-Yau manifold. Let us first assume that the Calabi-Yau is the total space of T^*M for some real 3-manifold M with $b_1(M) = 0$, and the lagrangian 3-cycle is the the base M. The open string field theory is now Chern-Simons on a real 3-manifold M. The open string algebra is given by $\mathcal{A} = \Omega^*(M, \mathrm{End}(E))$ for some flat gauge bundle $E \to M$, with $Q = d_A$ the covariant derivative and the product is again the wedge-product. Locally for the trivial bundle, we can approximate the open string algebra as the polynomial algebra $\mathbb{R}[x^\mu, \eta^\mu]$, with x^μ the coordinates of M and η^μ fermions of degree 1. As above, the Hochschild cohomology

$$\mathcal{C} = H^*_{\delta_m}(\mathrm{Hoch}(\mathcal{A})) \tag{9.1}$$

can then be approximated by the polynomial algebra $\mathcal{C} = \mathbb{R}[x^\mu, \eta^\mu; \chi_\mu, y_\mu]$, where χ_μ have degree 1 and y_μ have degree 0. The coboundary operator is given by

$$\delta_Q = \eta^\mu \frac{\partial}{\partial x^\mu} + \chi_\mu \frac{\partial}{\partial y_\mu}\,. \tag{9.2}$$

Again writing $\mathcal{C} = \mathbb{R}[x^\mu, \eta^\mu] \otimes \mathbb{R}[\chi_\mu, y_\mu]$, and first taking cohomology with respect to the second term $\chi_\mu \frac{\partial}{\partial y_\mu}$ in the coboundary δ_Q, the second factor has trivial cohomology. Therefore what remains is the first factor and the coboundary $\eta^\mu \frac{\partial}{\partial x^\mu}$, which is just the De Rham differential d on forms. Globally, this gives the De Rham cohomology

$$H^*(\mathrm{Hoch}(\mathcal{A})) = H^*(M)\,. \tag{9.3}$$

We can use Morita equivalence to argue that the Hochschild cohomology does not depend on the choice of flat gauge bundle E. As the Calabi-Yau manifold is contractible to M, this is the same as the cohomology of the full Calabi-Yau T^*M, which is the closed string algebra.

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In general this is however not precisely the closed string algebra. The latter is the cohomology of the Calabi-Yau space in which the 3-manifold M is embedded as a lagrangian cycle. We could have never found the full closed string, was the open string A-model as it stands knows only about the lagrangian cycle M. There is a way to understand this, by remembering that the A-model is a topological subsector of the superstring on the Calabi-Yau. The lagrangian submanifold M is the space of a D-brane. This theory however has as its low-energy degrees of freedom not only the gauge field, but also 3 scalars, which represent the position of the D-brane in the Calabi-Yau. It is through these scalars that the decoupled wrapped D-brane knows about the bulk space. The solution therefore is to include in the topological open A-model also these 3 scalars. This can be done as follows. Note that the scalars X together take values in the NM normal bundle to M in the Calabi-Yau. The Kähler form ω of the Calabi-Yau and the lagrangian condition of the cycle M defines an isomorphism of the normal bundle with the cotangent bundle of M, through $X \to \iota_X \omega$. We use this to define a complex gauge field $B = A + i\iota_X \omega$, in components $B_\mu = A_\mu + i\omega_{\mu i} X^i$. We propose a complex action which the usual Chern-Simons for this complexified gauge field. Note that because X is in the adjoint of the gauge group, this still is invariant under the usual gauge transformations $B \to U^{-1}BU + iU^{-1}dU$. The complex action is actually invariant under the full complexified gauge symmetry, with the same formula for the gauge transformations but U taking values in the complexified gauge group. The extra gauge transformations induce a nontrivial shift in the transverse coordinates, infinitesimally the new gauge transformations act on X as $\delta_\Lambda X^i = [X^i, \Lambda] + \omega^{i\mu}\partial_\mu \Lambda$. Looking at the U(1) sector, and interpreting the scalars as embedding coordinates of the lagrangian 3-brane, these shifts generate hamiltonian flows with respect to the Kähler structure. These are indeed natural candidates for gauge transformations of this system. The complexified action can be interpreted as a superpotential, as in the case of the B-model. This also makes it plausible that the action becomes complex. The complex EOM found from this action are then interpreted as F-terms. Let us concentrate on the abelian model. The real part of this F-term is the usual condition for a flat connection $F = dA = 0$. The imaginary part becomes $d(\iota_X \omega) = \mathcal{L}_X \omega = 0$. This is actually equivalent to the condition that the 3-cycle shifted by the normal coordinates X^i is still lagrangian.

To see that this can give the correct Hochschild cohomology we consider the case of a torus fibration over M. Let us assume that the total CY space looks like T^*M with a nontrivial identification by a lattice $\Lambda \subset \mathbb{R}^3$ in the fiber, where we identified the fiber with $\mathbb{R}^3$. This implies that we have an identification of the scalars $X^i \to X^i + v^i$, for $v \in \Lambda$. As we understand $iX^i\omega_{i\mu}\eta^\mu$ as an element of the algebra $\mathcal{A}$ it means that elements of this algebra should be identified modulo elements of the form $iv^i\omega_{i\mu}\eta^\mu$ for $v \in \Lambda$. For consistency, also products of such elements have to be identified with zero. After choosing coordinates such that $\omega\Lambda$ is identified with $\mathbb{Z}^3$ we can write these trivial elements as an integral subalgebra $i\mathbb{Z}[\eta^\mu] \subset \mathbb{C}[x^\mu, \eta^\mu]$. The identification of elements in $\mathcal{A}$ modulo this subalgebra means that we should replace the open string algebra by the quotient algebra $\mathbb{C}[x^\mu, \eta^\mu]/i\mathbb{Z}[\eta^\mu]$.

To see the effect of a quotient on the Hochschild cohomology, let us first consider the most trivial case where we have a single anticommuting generator η and take the quotient algebra $\mathcal{A} = \mathbb{R}[\eta]/\mathbb{Z}[\eta]$. We then approximate the Hochschild cohomology $HH(\mathcal{A})$ using the HKR theorem introducing the dual variable $y \sim \frac{\partial}{\partial\eta}$. We still have to divide by the discrete algebra, so we have $\mathbb{R}[\eta, y]/\mathbb{Z}[\eta]$. We should require that the action of any element in the Hochschild complex acts trivially on the subalgebra $\mathbb{Z}[\eta]$ that we divided out. Taking the element $\mathrm{e}^{n\eta} \in \mathbb{Z}[\eta]$ for $n \in \mathbb{Z}$, we find

$$\phi(\eta, y)\mathrm{e}^{n\eta} = \mathrm{e}^{n\eta}\phi(\eta, y+1)\,. \tag{9.4}$$

This shows that the dual variable y must be periodic. More generally, when we have several generators η_μ and consider a quotient algebra $\mathbb{R}[\eta_\mu]/\Lambda[\eta_\mu]$, where Λ is a lattice and $\Lambda[\eta_\mu]$ is the subalgebra degenerated by elements of the form $v^\mu\eta_\mu$ for $v \in \Lambda$, the Hochschild cohomology is described as the function algebra $\mathbb{R}[y^\mu, \eta_\mu]$ where y^μ are coordinates on a periodic plane with periods given by Λ, $y \sim y + v$.

Going back to our original problem, we find that the dual generators y_μ in the Hochschild cohomology are periodic with according to the identification $y_\mu \to \omega_{\mu i}v^i$. This suggests that we should understand $x^i = \omega^{i\mu}y_\mu$ as the transverse coordinates on the total Calabi-Yau 3-fold. If we also understand $\eta^i = \omega^{i\mu}\chi_\mu$ as vertical 1-forms dx^i, we see that the Hochschild consists of all the (complex) forms on the full Calabi-Yau. Taking into account the BRST operator Q, we see that δ_Q is identified with the De Rham differential d of the CY. This also shows that the total cohomology of the Hochschild double complex is precisely the complexified De Rham cohomology of the total space.

9.3 The C-model

Similarly, we can do the same for the Cattaneo-Felder model [19], or C-model. The open string algebra is the algebra of functions $\mathcal{A} = C(M)$ on a manifold M. The BRST operator is zero, and the product is the usual product of functions. Approximating functions by polynomials, we have $\mathcal{A} = \mathbb{C}[x^\mu]$. Therefore the HKR theorem gives

$$\mathcal{C} = H^*_{\delta_m}(\mathrm{Hoch}(\mathcal{A})) = \mathbb{C}[x^\mu, \chi_\mu]\,, \tag{9.5}$$

where the generators χ_μ have degree 1. This generates the algebra of polyvector fields $\Gamma(M, \bigwedge^* TM)$, by identifying the χ_μ with the basis of vector fields ∂_μ. Because $Q = 0$, this already equals the full Hochschild cohomology $H^*(\mathrm{Hoch}(\mathcal{A}))$. Indeed, this is the on-shell closed string algebra of the C-model.

The relation of this model to the deformation theory of the algebra of functions, through the Hochschild complex, was the main topic of the original paper [19]. Given a bulk operator ϕ (called α in the paper) related to a polyvector field, they constructed polydifferential operators $U(\phi) : C^\infty(M)^{\otimes n} \to C^\infty(M)$, defined through the path integral,

$$U(\phi)(f_1, \cdots, f_n)(x) = \left\langle \delta_x(X)(f_1(X)\cdots f_n(X))^{(n-2)} \right\rangle_\phi\,, \tag{9.6}$$

where the subscript ϕ indicates that the action is shifted by the term canonically related to ϕ, and δ_x is the delta function at the point x. We see that the $U(\phi)$ can be identified with our $\Phi^{(n)}_\phi$ to first order in ϕ.

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10. Conclusion and discussion

In this paper we have calculated mixed correlation functions in the topological B-model open-closed string theory. Our most important goal was to understand these in terms of deformations of the open string theory. These give the natural map from closed string BRST cohomology to the Hochschild cohomology of the open string algebra. By an explicit calculation of this Hochschild cohomology we found that in fact this map is an isomorphism. We conjectured in the introduction that this is true more generally, for any 2-dimensional topological field theory. In this paper we also checked this conjecture for the A-model in certain simple situations. One problem with this general conjecture is that it is not obvious what the closed string theory is given the open string sector. The conjecture could perhaps better be interpreted as a canonical construction of the latter.

Although this calculation of the total cohomology of the Hochschild complex corresponds to the closed string BRST cohomology, we saw that the intermediate Hochschild cohomology $HH^*(\mathcal{A}) = H^*_{\delta_m}(\mathrm{Hoch}(\mathcal{A}))$ contained all the closed string operators. More generally, the BV structure of this algebra allowed us to reproduce the full closed string B-model. Explicitly, we used the data to write down the BV sigma model for the B-model.

In this paper we have considered mainly the B-model for a single D-brane. In general we can have bound states of several branes. These bound states can be considered as objects in the derived category of holomorphic sheaves $\mathcal{D}(M)$ [58]. More generally, we can consider the full category of these boundary conditions at once. This catagory has an A_∞ structure; though in a categorical sense. The Hochschild cohomology of $\mathcal{D}(M)$ is in fact the Dolbeault cohomology, $H^{*,*}(M) \cong HH^*(\mathcal{D}(M), \mathcal{D}(M))$. Also from this point of view the Frobenius structure of the closed string (the closed string action of [25]) is reproduced, see e.g. [59]. The fact that in general we should consider more objects at once also becomes clear when one looks at a cycle. In order to reproduce the full closed string, one needs at least a family of embedding curves. This became more important for the A-model, where all open string models are based on a cycle of lower dimension. This construction has an obvious relation, and indeed was inspired by, homological mirror symmetry [60, 61, 62, 63].

Another manifestation of this correspondence is related to the homology of loop space or string homology. It was shown [64] that it equals the Hochschild cohomology of the cochain complex $C^*(M)$, that is $H_*(LM) \cong HH^*(C^*(M), C^*(M)^*)$. The loop space has an obvious relation with the closed string. This should actually be understood in terms of the C-model, where the full open string algebra indeed is the algebra of differential forms [19].

It would be very interesting to see if our conjectured relation between the Hochschild cohomology of the open string and the on-shell closed string algebra holds true in the case of the bosonic string. It is known that the bosonic open string field theory can be formulated in terms of a differential graded algebra. This can be very interesting in the context of tachyon condensation [65]. At this point, the open string field theory should describe the closed string vacuum, where only the closed strings are present. Note that this example shows the importance of using the full off-shell open string algebra, rather than its cohomology. Indeed, the cohomology in this case is trivial, while we expect a nontrivial

Hochschild cohomology. There are several problems however checking the proposal in this case. First of all, the (off-shell) open string algebra is very big, consisting of an infinite number of field components. Furthermore, the product structure, defined in terms of gluing half-strings, is much more complicated than the ones we saw in this paper. It can not be formulated in terms of the product in a polynomial or function algebra, therefore we can not use the HKR theorem.

It was recently argued that in vacuum string field theory the (first order) coupling between the open and closed string is linear in the open string field [7, 8]. If our conjecture about the Hochschild cohomology holds true in VSFT this would imply that, at least with respect to a suitable basis, the Hochschild cohomology is completely concentrated in degree zero. Recently a much simpler expression for the star product was developed in [66]. In this paper it was shown that, at least in the zero momentum sector, the open string star product reduces to a continuous product of Moyal products. This would reduce the calculation of the second term in the spectral sequence, the cohomology $HH^*(\mathcal{A})$ with respect to δ_m, to that of the ordinary star product on the noncommutative plane $\mathbb{R}^2_\theta$ for varying θ. The latter cohomology is however trivial; it lives only in degree zero. This corresponds to operators which are linear in the open string field. This could therefore be an explanation why in open string field theory the coupling to the closed string should be linear.

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A. Gerstenhaber structure of the Hochschild cohomology

A.1 Gerstenhaber algebras and BV algebras

A *Gerstenhaber algebra* is a $\mathbb{Z}$-graded algebra with a graded commutative associative product $\cdot$ of degree 0 and a bracket $[\cdot,\cdot]$ of degree -1 (the Gerstenhaber bracket), which is such that $A[1]$ is a graded Lie algebra. This implies that it is twisted graded antisymmetric,

$$[\alpha,\beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,\alpha]\,, \tag{A.1}$$

and satisfies a twisted graded Jacobi identity,

$$[\alpha,[\beta,\gamma]] = [[\alpha,\beta],\gamma] + (-1)^{(|\alpha|-1)(|\beta|-1)}[\beta,[\alpha,\gamma]]\,. \tag{A.2}$$

Furthermore, the map $[\alpha,\cdot]$ must be a graded derivation of the product,

$$[\alpha,\beta\cdot\gamma] = [\alpha,\beta]\cdot\gamma + (-1)^{(|\alpha|-1)|\beta|}\beta\cdot[\alpha,\gamma]\,. \tag{A.3}$$

We can generalize this to a *differential Gerstenhaber algebra* (or *DG*) by adding a differential δ of degree 1, satisfying the graded derivation conditions with respect to the product and the bracket. We note that these identities are twisted graded variants of a *Poisson algebra*.

A *BV algebra* is a Gerstenhaber algebra with a bracket of degree 1, (called the BV bracket) supplied with a nilpotent degree 1 operator $\triangle$ (the BV operator), $\triangle^2 = 0$, such that the bracket is given as the failure of its derivation property,

$$[\alpha, \beta] = (-1)^{|\alpha|}\triangle(\alpha \cdot \beta) - (-1)^{|\alpha|}\triangle(\alpha) \cdot \beta - \alpha \cdot \triangle\beta\,. \tag{A.4}$$

A.2 Hochschild complex and cohomology

The Hochschild complex of an associative algebra A is defined in terms of the space of multilinear maps $C^n(A, A) = \mathrm{Hom}(A^{\otimes n}, A)$. The coboundary of this complex is given in terms of the product $m = \cdot$ of A. For $\phi \in C^n(A, A)$ it is defined by

$$\begin{aligned}\delta_m \phi(\alpha_0, \cdots, \alpha_n) \;=\; & -\alpha_0 \cdot \phi(\alpha_1, \cdots, \alpha_n) + (-1)^n \phi(\alpha_0, \cdots, \alpha_{n-1}) \cdot \alpha_n + \\ & + \sum_{i=0}^{n} (-1)^i \phi(\alpha_0, \cdots, \alpha_i \cdot \alpha_{i+1}, \cdots, \alpha_n)\,. \end{aligned} \tag{A.5}$$

It is well known that the Hochschild complex is endowed with a natural Gerstenhaber bracket. We first define a composition of two elements $\phi_i \in C^{n_i}(A, A)$ by

$$\phi_1 \circ \phi_2(\alpha_1, \ldots, \alpha_{n_1+n_2-1}) = \sum_i (-1)^{\epsilon_i} \phi_1(\alpha_1, \ldots, \alpha_i, \phi_2(\alpha_{i+1}, \ldots, \alpha_{i+n_2}), \ldots, \alpha_{n_1+n_2-1})\,. \tag{A.6}$$

where $\epsilon_i = (n_2 - 1)i + \sum_{k=1}^{i} |\phi_2||\alpha_k|$. The Gerstenhaber bracket can now be defined as the graded commutator of this composition

$$[\phi_1, \phi_2] = \phi_1 \circ \phi_2 - (-1)^{(n_1-1)(n_2-1)+|\phi_1||\phi_2|} \phi_2 \circ \phi_1\,. \tag{A.7}$$

This can be interpreted as a double graded supercommutator, with the degrees $(n-1, |\phi|)$. These are indeed the natural gradings in the Hochschild complex. In additional there is a natural cup-product, defined by

$$\phi_1 \cup \phi_2(\alpha_1, \cdots, \alpha_{n_1+n_2}) = \phi_1(\alpha_1, \cdots, \alpha_{n_1}) \cdot \phi_2(\alpha_{n_1+1}, \cdots, \alpha_{n_1+n_2})\,. \tag{A.8}$$

On cohomology, the cup-product can be shown to be symmetric. Together, the bracket and the cup product it makes the Hochschild cohomology $HH^*(A, A) = H^*_{\delta_m}(C^*(A, A))$ into a Gerstenhaber algebra.

Note that if we interpret the product as an element $m \in C^2(A, A)$, we can write the action of the Hochschild coboundary on $\phi \in C^n(A, A)$ as $\delta_m \phi = (-1)^n [m, \phi]$. If A would be a differential graded algebra, with differential Q of degree 1, we could similarly introduce a second coboundary by the relation $\delta_Q \phi = [Q, \phi]$. This makes the Hochschild complex of a differential graded algebra into a double complex, with the natural bigrading mentioned above (such that δ_m has bidegree $(1, 0)$ and δ_Q has bidegree $(0, 1)$). More general, when there is a full A_∞ structure with multiliner maps m_k, we have the differentials δ_{m_k} defined on $\phi \in C^n(A, A)$ by $\delta_{m_k} \phi = (-1)^{(k-1)n} [m_k, \phi]$. The A_∞ relations are equivalent to the nilpotency of the total coboundary $\sum_k \delta_{m_k}$.

A.3 The Hochschild cohomology of a polynomial algebra

We can give an explicit description of the Hochschild cohomology of a general polynomial algebra. Consider the algebra of polynomials in a finite number of $\mathbb{Z}$-graded variables x^i of degree $|x^i| = q_i \in \mathbb{Z}$, so the space $\mathcal{A} = \mathbb{C}[x^1, \ldots, x^N]$. The Hochschild cohomology of this algebra is, as a $\mathbb{Z}$-graded vector space, the algebra of polynomials $HH^*(\mathcal{A}) = \mathbb{C}[x^1, \ldots, x^N, y_1, \ldots, y_N]$ in the doubled set of variables x^i, y_i, where the extra generators have degree $|y_i| = 1 - q_i$ [15]. The Gerstenhaber bracket of this polynomial algebra is given by $\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_i}$. A representative of the cohomology class of any element of the cohomology in the Hochschild complex can be given by the polydifferential operator obtained by replacing y_i with $\frac{\partial}{\partial x^i}$ (all differentiations acting on different arguments). For example $\theta(x,y) = \theta^{ij}(x) y_i y_j \in HH^*(\mathcal{A})$ has as a representative the bidifferential operator $\theta^{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. This result is known as the Hochschild-Kostant-Rosenberg theorem. More generally, for the algebra $\mathcal{A} = \mathcal{O}(M)$ of regular functions on a smooth $\mathbb{Z}$-graded algebraic supermanifold M, the Hochschild cohomology is given by the algebra of functions on the total space of the twisted by [1] cotangent bundle to M, $HH^*(\mathcal{O}(M)) = \mathcal{O}(T^*[1]M)$. Because the new space is a cotangent bundle, it has a natural Gerstenhaber bracket, where the degree of -1 is a result of the twisting.

There is also an analytic version of this theorem, which is due to Connes.

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